On the Estimation of Parameters of Nonlinear Model in the Presence of Variance Homogeneity and Variance Heterogeneity.

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ABSTRACT

In this study, nonlinear model with variance homogeneity is compared with nonlinear model with variance heterogeneity (power-of-the-mean-variance model) using residual standard error and F-statistic to see which one gives parsimonious description of the datasets. Newton-Raphson Algorithm was used to estimate the parameters of the models. The two models are fitted to Carbon Monoxide (CO) pollution data measured in part per million (PPM). Based on residual standard error and F-statistic, the power-of-the-mean-variance model performed better than nonlinear model with variance homogeneity.

(Keywords: variance, homogeneity, heterogeneity, parameters, Newton-Raphson algorithm)

INTRODUCTION

Ratkowsky (1990) provides a very helpful catalogue of nonlinear mean functions that can be used in practice to model datasets that are nonlinear in nature. Among the functional forms available for modelling nonlinear relationships can be found in Bates and Watts (1998), Carroll and Ruppert (1988), Drapper and Smith (1996), Huet, et al. (2004), Ripley and Venables (2005), Seber and Wild (1988), and Ritz and Strreibig (2008).

Ratkowsky (1990) points out that none of these parametrizations dominate others with regard to computational and asymptotic properties. This means that other parametrizations of the same function may be used in some circumstances.
Consider any nonlinear mean function

(say) $y_i = f(x_i, \theta)$, the error term (ε_i)

will always be introduced into the mean function be either multiplicatively, additively or otherwise. The essence of the error term (ε_i) is to make the model a statistical model and also to cater for the distortion in the response y_i away from expected value $f(x_i, \theta)$ caused by various unknown sources of variation and the error would be varies from measurement to measurement.

Typically, the error term (ε_i) are assumed to be normally distributed (with mean 0 and variance σ^2) i.e., assumptions underlying nonlinear regression model. In this paper we proposed an approach for dealing with model violation related to the measurement error: normally distributed with heterogeneous variance.

Carroll and Ruppert (1988; pp51-61) pointed out that variance heterogeneity has little influence on the parameter estimates θ, but if ignored it may result in severely misleading confidence and prediction intervals. However, one way of taking into account variance heterogeneity is by explicitly modelling it. Although, variance modelling will only cater for variance heterogeneity; but it is not a remedy for non-normal errors. Meanwhile, both non-normal error distributions and variance heterogeneity can be adequately tackled by transformation approach.

MATERIALS AND METHODS

The relationship between response and predictor variables can be formulated as:

$$y = f(x; \theta) \qquad (1)$$

Where f (.) would be non linear in one or more of the p parameters $\theta_1 \ldots \theta_p$. The parameters have to be estimated from the data. However, the number of parameters occurring in **f** should be less than the number of observations (i.e. $p<n$). The relationship in Equation 1 is for the ideal situation, the predictor values $x_i, \ldots, x_n$ and the response $y_i, \ldots, y_n$ are observed without error. In reality, measurement error will distort the picture such that none of pairs $(\mathrm{x}_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ will fit Equation 1, exactly.

Nonlinear Model with Variance Homogeneity

Here, we assumed that the error term enters the model additively; therefore, the complete specification of the nonlinear regression model is given as:

$$\left.\begin{array}{r} y_i = f(x_i;\theta) + \varepsilon_i \\ E(\varepsilon_i) = 0; v(\varepsilon_i) = \sigma^2 \end{array}\right\} \qquad (2)$$

In matrix form, it can be expressed as:

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \quad x = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}, \quad \theta = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_p \end{bmatrix}, \quad \varepsilon = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix}$$

$$(\varepsilon_t^1 \varepsilon_8) = \begin{cases} 0 & t \neq 8 \\ \sigma^2 & t = 8 \end{cases}$$

Estimating the Parameters of Nonlinear System with Variance Homogeneity

Nonlinear least squares and maximum likelihood are the most common methods of parameter estimation in nonlinear models. Both methods involve finding the value that minimizes or maximizes a function. In this paper, we employed the method of nonlinear least squares.

Consider the Equation 2, the problem is to estimate the unknown vector θ, a natural solution is to choose the value of θ that minimizes the distances between the values of $f(x_i;\theta)$ and observation y_i; this is referred as the least squares criterion or the residual sums of squares (RSS) with respect to θ. This is defined as:

$$Q = RSS(\theta) = \sum_{i=n}^{n} (y_i - f(x_i;\theta))^2 \qquad (3)$$

The partial derivative of $RSS(\theta)$ with respect to θ is given as:

$$\frac{\partial Q}{\partial \theta} = -2\sum_{i=n}^{n} [y_i - f(x_i;\theta)] = \left[\frac{\partial f(x_i;\theta)}{\partial \theta_a}\right] \qquad (4)$$

Where $a = 1, \ldots, p-1,$, when the p partial derivatives are each set to 0, we obtain thereafter the p-normal equations after some simplification; compactly, it can be written as;

$$\sum_{i=n}^{n} y_i \left[\frac{\partial f(x_i;\theta)}{\partial \theta_a}\right] - \sum_{i=n}^{n} \left[\frac{\partial f(x_i;\theta)}{\partial \theta_a}\right] [f(x_i;\theta)] = 0 \qquad (5)$$

There is no explicit solution to Equation (5). It is mathematically intractable. It poses serious computational task, the estimates can be obtained using numerical minimization. However, a variety of reliable algorithms are now available to overcome these difficulties. We therefore employed Gauss-Newton Algorithm.

Gauss- Newton Algorithm for Fitting Nonlinear Models

The Gauss-Newton Algorithm requires the calculation of first derivatives. If the residuals are small, the Gauss-Newton algorithm will converge more rapidly. Suppose the model to be fitted is:

$$\left.\begin{array}{r} y_i = f(x_i;\theta) + \varepsilon_i \\ E(\varepsilon_i) = 0; v(\varepsilon_i) = \sigma^2 \end{array}\right\}$$

The model is that $f(x_i;\theta)$ is assumed to be nonlinear model with mean zero and unknown variance σ^2. The sum of squared departures is:

$$s(\theta) = \sum_{i=n}^{n} [y_i - f(x_i;\theta)]^2 \qquad (6)$$

These estimates by definition minimize S (θ), therefore, the least squares estimate of θ is obtained by differentiating equation (3) with respect to θ and equate to zero and solve for:

$$\sum_{i=n}^{n} [y_i - f(x_i;\theta)] = \left[\frac{\partial f(x_i;\theta)}{\partial \theta_j}\right] = 0$$
$$i = 1,2,\ldots,n, \quad j = 1,2,\ldots,j \qquad (7)$$

Gauss-Newton algorithm starts by expanding the deterministic component $f(x_i;\theta)$ of Equation (2) using Taylor first-order approximation $f(x_i;\theta)$, we have:

$$f(x,\theta) = f(x,\theta^0) + \left[\frac{\partial f(x_i;\theta)}{\partial \theta_j}\right](\theta - \theta_0) | \, \theta = \theta_0 \qquad (8)$$

Substitute Equation (9) into Equation (1):

$$y_i = f(x,\theta^0) + \left[\frac{\partial f(x_i;\theta)}{\partial \theta_j}\right](\theta - \theta_0) \; | \, \theta = \theta_0 + \varepsilon_i \qquad (9)$$

On- rearranging, the Equation 10 becomes:

$$y_i - f(x,\theta^0) + \frac{\partial f(x_i;\theta)}{\partial \theta_j}(\theta - \theta_0) \; | \, \theta = \theta_0 + \varepsilon \qquad (10)$$

Let $f_i^o = f(x;\theta)$ then:

$$y_i - f_i^o(x,\theta^0) = \left[\frac{\partial f(x_i;\theta)}{\partial \theta_j}\right](\theta - \theta_0) \; | \, \theta = \theta_0 + \varepsilon_i \qquad (11)$$

Equation 12 can be interpreted as follow; for the j^{th} parameters and i^{th} observations, we have:

$$y_i - f_i^o = \left[\frac{\partial f(x_1,\theta)}{\partial \theta_1}\right](\theta_1 - \theta_1^{\,0}) + \left[\frac{\partial f(x_2,\theta)}{\partial \theta_2}\right](\theta_2 - \theta_2^{\,0}) + \cdots + \left[\frac{\partial f(x_i,\theta)}{\partial \theta_j}\right](\theta_j - \theta_j^{\,0}) + \varepsilon_i$$

It can be written as:

$$y_i - f_i^o = \sum_{i=1}^{j} \left[\frac{\partial f(x_i,\theta)}{\partial \theta_j}\right](\theta_j - \theta_j^{\,0}) + \varepsilon_i \qquad (12)$$

Let:

$\beta_j = \theta_j - \theta_j^0$ and $\quad z_{ji} = \dfrac{\partial f(x_i,\theta)}{\partial \theta_j} | \, \theta - \theta_0 \qquad (13)$

Consider the n-observations:;

$$\begin{aligned}
y_1 - f_1^0 &= \beta_1 Z_{11} + \beta_2 Z_{21} + \ldots + \beta_j Z_{j1} + \varepsilon_1 \\
y_2 - f_2^0 &= \beta_2 Z_{22} + \beta_2 Z_{22} + \ldots + \beta_j Z_{j2} + \varepsilon_2 \\
&\vdots \\
y_n - f_n^0 &= \beta_1 Z_{1n} + \beta_2 Z_{2n} + \ldots + \beta_j Z_{jn} + \varepsilon_n
\end{aligned}$$

$$y_i - f_i^0 = \beta_1 Z_{11} + \beta_2 Z_{21} + \varepsilon_i$$

$$\begin{bmatrix} y_1 - f_1^0 \\ y_2 - f_2^0 \\ \vdots \\ y_n - f_n^0 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{21} & \cdots & Z_{1n} \\ Z_{12} & Z_{22} & \cdots & Z_{2n} \\ \vdots & & \ddots & \vdots \\ Z_{j1} & Z_{j2} & \cdots & Z_{jn} \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_j \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_j \end{bmatrix}$$

In matrix form; we have;

$$y_i - f^0 = Z^0 \beta^0 + \varepsilon_i \qquad (14)$$

The least squares of β^0 can therefore be written as:

$$\beta^0 = \left(Z^{01} Z^0\right)^{-1} Z^0 \left(Y - f^0\right) \qquad (15)$$

We have minimized the sum of squares, however, the residual sum of squares can be written as:

$$S(0) = \sum_{i=1}^{N}\left[y_i - f_i^0 - \sum_{i=1}^{n} \beta_i^0 Z_{ji} \right]^2 \qquad (16)$$

Recalling that:

$$\beta_j = \theta_j - \theta_j^0$$

As yielded by equation (16), the latest estimate of β_j is:

$$\theta_j = \beta_j + \theta_j \qquad (17)$$

Recalling from equation (16):

$$\hat{\beta} = \left(Z^{01} Z^0\right)^{-1} Z^0 \left(Y - f^0\right)$$
$$\theta^{r+1} = \left(Z^{01} Z^0\right)^{-1} Z^0 \left(Y - f^0\right) + \theta^r$$
$$\theta^{r+1} = \beta^r + \theta^r$$

Thus:

$$\beta = \theta^{r+1} - \theta^r$$

The iterative process continues until convergence is achieved, convergence is achieved at r+1 iteration.

Termination Condition of Iteration

The condition for convergence is given by:

$$\left| \frac{\theta^{r+1} - \theta^r}{\theta^r} \right| < \delta$$

Where δ is some pre-determined small amount [e.g. 0.000001 or $\left(1.0 \times 10^{-6}\right)$]. At the end of $(r+1)^{th}$ iteration procedure, the estimates of S(θ) is defined by:

$$S(\theta) = S(0) = \sum_{i=1}^{n}\left[y_i - f_i^r - \sum_{i=1}^{n} \beta^r z^r ji \right]^2$$

Also, $S(\theta)$ can be checked to see if a reduction in its value has actually been attained.

Estimate of Error Term Variance

An estimate of the error term variance σ^2 can be obtained for nonlinear regression model in the same form for linear regression model:

$$MSE = \frac{SSE}{n-p} = \frac{\sum_{i=1}^{n} \left[\left(y_i - f(x_i;\theta)\right)\right]^2}{n-p} \qquad (18)$$

MSE is not an unbiased estimator of σ^2, but the bias is small when the sample size is large.

Nonlinear Model with Variance Heterogeneity

We have considered model with additive error term, where we assumed that all errors had the same variances $\operatorname{var}(\varepsilon_i)$. This implying that the errors were identically distributed. Here, we want to consider the mean Variance-modelling in an attempt to deal with model violation related to the measurement error-normally distributed with heterogeneous variance, also for data characterized by various heterogeneity, a situation where the variance changes as the predictor values changes.

The model is specified as:

$$\left.\begin{array}{l} y_i = f(x_i;\theta) + \varepsilon_i \\ E(\varepsilon_i) = 0; \operatorname{var}(\varepsilon_i) = \sigma^2\left(f(x_i;\theta)\right) \end{array}\right\} \qquad (19)$$

However, for an increasing variance, the way to model the dependence of the variance on the a mean is through power-of-the-mean-variance-model:

$$\mathrm{var}(\varepsilon_i) = \sigma^2 \left(f(x_i;\theta) \right)^{2\delta}$$

The variance of each observation depends on the corresponding mean value $f(x_i;\theta)$ through power function with exponent 2δ. Variance homogeneneity corresponding to $\delta = 0$ in which case (20) becomes (2). This means that the model with variance homogeneity is a sub-model of the power-of-the-mean-model (variance homogeneity's model) and, consequently, a statistical test F-test can be used to assess whether or not the power-of-mean-variance-model provides a significant improved fit compared with model that assumes variance homogeneity.

Estimating the Parameters of Nonlinear System with Variance Heterogeneity

There are different methods of carrying out the parametric modelling of the variance function. In this paper, the method of maximum likelihood is choosing because it is widely known. Consider the nonlinear regression model:

$$\left. \begin{array}{l} y_i = f(x_i;\theta) + \varepsilon_i \\ E(\varepsilon_i) = 0; \mathrm{var}(\varepsilon_i) = \sigma^2 \left(f(x_i;\theta) \right) \end{array} \right\}$$

The likelihood function is given as:

$$L(\alpha, \beta, \sigma^2) = \frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left[\frac{1}{2\sigma^2} \sum_{i=1}^{n} (y_i - \alpha e^{x/\beta})^2 \right] \qquad (24)$$

$$h(\alpha, \beta) = \sum_{i=1}^{n} [(y_i - \alpha \exp(x/\beta))]^2$$

$$h(\alpha, \beta) = \sum_{i=1}^{n} \left[\left(y_i - \alpha \exp\left(\frac{x}{\beta} \right) \right) \right]^2 \frac{\partial f(x, \alpha, \beta)}{\partial \alpha, \beta}$$

$$h(\alpha, \beta) = \sum_{i=1}^{n} \left[\left(y_i - \alpha \exp\left(\frac{x}{\beta} \right) \right) \right]^2 \frac{\partial f(x, \alpha, \beta)}{\partial \alpha, \beta} = -2\sum_{i=1}^{n} \left[\left(y_i - \alpha \exp\left(\frac{x}{\beta} \right) \right) \right] \left[\exp\left(\frac{x}{\beta} \right) \right]$$

$$h(\alpha, \beta) = \sum_{i=1}^{n} \left[\left(y_i - \alpha \exp\left(\frac{x}{\beta} \right) \right) \right]^2 \frac{\partial f(x, \alpha, \beta)}{\partial \alpha, \beta} = -2\sum_{i=1}^{n} \left[\left(y_i - \alpha \exp\left(\frac{x}{\beta} \right) \right) \right] \left[\frac{x\alpha \exp(x/\beta)}{\beta^2} \right]$$

The normal equations are;

$$\sum_{i=1}^{n} \left[\left(y_i - \alpha \exp\left(\frac{x}{\beta} \right) \right) \right] \left[\exp\left(\frac{x}{\beta} \right) \right] = 0 \qquad (25)$$

$$\sum_{i=1}^{n} \left[\left(y_i - \alpha \exp\left(\frac{x}{\beta} \right) \right) \right] \left[\frac{x\alpha \exp(x/\beta)}{\beta^2} \right] = 0 \qquad (26)$$

There is no closed form solution; the normal equations are mathematically intractable.

Let:

$$Zji = \frac{\partial f(x;\theta)}{\partial \theta_j}; \theta_j - \theta_j$$

$$Zj1 = \frac{\partial f(x;\alpha,\beta)}{\partial \alpha} = \exp\left(\frac{x}{\beta}\right)$$

$$Zj2 = \frac{\partial f(x;\alpha,\beta)}{\partial \alpha} = \frac{x\alpha \exp(x/\beta)}{\beta^2}$$

Consider n-observations:

$$y_i - f_i = \beta_1 z_{11} + \beta_2 z_{21} + \mu_i$$

$$f_i = \alpha \exp\left(\frac{x}{\beta}\right); zj1 = \exp\left(\frac{x_i}{\beta}\right); zj2 = \frac{x\alpha \exp(x/\beta)}{\beta^2}$$

$$y_i - \alpha \exp\left(\frac{x}{\beta}\right) = \beta_1 \exp\left(\frac{x_i}{\beta}\right) + \beta^2 \frac{x_1\alpha \exp(x/\beta)}{\beta^2} + \mu_i$$

In matrix form:

$$\begin{bmatrix} y_1 - \alpha \exp\left(\frac{x_1}{\beta}\right) \\ y_2 - \alpha \exp\left(\frac{x_2}{\beta}\right) \\ \vdots \\ y_n - \alpha \exp\left(\frac{x_n}{\beta}\right) \end{bmatrix} = \begin{bmatrix} \exp\left(\frac{x_1}{\beta}\right) & \frac{x_1\alpha \exp(x_1/\beta)}{\beta^2} \\ \exp\left(\frac{x_2}{\beta}\right) & \frac{x_2\alpha \exp(x_2/\beta)}{\beta^2} \\ \vdots & \vdots \\ \exp\left(\frac{x_n}{\beta}\right) & \frac{x_n\alpha \exp(x_n/\beta)}{\beta^2} \end{bmatrix} \begin{bmatrix} \gamma_1^0 \\ \gamma_2^0 \end{bmatrix} + \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_n \end{bmatrix}$$

In matrix form:

$$y_i^0 - f_i^0 = z^0\gamma^0 + \mu_i$$

The least square of γ^0 can be written as:

$$\gamma^0 = \left(z^0 z^0\right)^{-1} z^0\left(y_i^0 - f_i^0\right)$$

We have minimized the sum of square $S(\theta)$

Estimate of Error Term Variance

An estimate of the error term variance σ 2 is given as;

$$MSE = \frac{SSE}{n-p} = \frac{\sum_{i=1}^{n} [(y_i - f(x_i;\alpha,\beta))]^2}{n-p} \qquad (27)$$

Here α and β are the final parameter estimates, the residuals are the deviations around the fitted nonlinear regression function using the final estimated coefficients.

Maximum Likelihood Estimation

Consider the nonlinear regression model:

$$\begin{aligned} &Y_i = f(x_i;\alpha,\beta) + \varepsilon_i \\ &Var(\varepsilon_i) = \sigma^2(x_i;\alpha,\beta,\delta), E(\varepsilon_i) = 0 \end{aligned}$$

The likelihood function is given as:

$$\ell(y,x,\alpha,\beta,\delta,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2 g(x_i,\alpha,\beta,\delta)}} \exp\left(-\frac{\sum_{i=1}^{n} (y - f(x,\alpha,\beta))^2}{2\sigma^2 g(x,\alpha,\beta,\delta)} \right) \qquad (28)$$

For ease of handling, we generally use the logarithm of the likelihood:

$$V(y,x,\theta,\sigma^2,\beta,\delta) = \frac{n}{2}\log(2\pi) \ \frac{1}{2}\sum_{i=1}^{n} \left(\log\sigma^2(x_i,\alpha,\beta,\delta) + \frac{(Y_i - f(x_i,\alpha,\beta))^2}{2\sigma^2(x_i,\alpha,\beta,\delta)} \right) \qquad (29)$$

For a given set of observations $Y_1, \ldots, Y_n$, the log-likelihood is a function of the parameters. If p is the dimension of (α, β) and q is the dimension of β, the model depends on p+q+1 parameters.

The maximum likelihood estimator $\hat{\alpha}, \hat{\theta}, \hat{\sigma}^2, \hat{\delta}$ maximizes the log-likelihood. The values of $\hat{\alpha}, \hat{\theta}, \hat{\sigma}^2, \hat{\delta}$ cannot be obtain explicitly; they obtained by numerical computation.

Using F-Test

To obtain the most parsimonious description of the dataset, a statistical test (F-test) is carried out between the two models. This is necessary because two models are fitted: Model A and Model B. Model A is a sub-model or nested model of Model B.

The F-test is defined as:

$$F = \frac{\left(RSS_A\left(\hat{\theta}_A\right)\right) - \left(RSS_B\left(\hat{\theta}_B\right)\right) / dfA - dfB}{SS_B / df_B}$$

Where subscripts A and B refer to model A and model B, respectively.

Model A − Model with variance homogeneity (sub model of Model B)

Model B− Model with variance heterogeneity (Power-of-mean-variance).

The main ingredient in Equation 28 is the different between RSS quantities for the two models considered. A large difference means that the two models are quite different, whereas a small difference indicates that they provide similar fits to the data. Large and small can be assessed and quantified by means of a p-value obtained from an F-distribution with degrees of freedom $dfB - dfA, dfA$.

DATA ANALYSIS AND DISCUSSION

Table 1: Summary of Exponential Model with Variance Homogeneity.

Parameters	Estimate	Std. Error	t-value	$pr(>\|t\|)$
α	4.7893	0.59826	8.006	8.29e-14
β	24.8589	0.8998	27.627	2e-16

The Residual Standard Error is 23.57 on 208 degree of freedom and number of iteration to convergence is 4. Achieved convergence tolerance is 2.273e-06, while LogLik is -960.5496.

Table 2: Summary of Exponential Model with Variance Heterogeneity.

Parameters	Estimate	Std. Error	t-value	$pr(>\|t\|)$
α	3.834931	0.3719501	10.31034	0
β	23.301546	0.6610329	35.25020	0

The variance parameter $\hat{\delta}$ = 0.8604619 very close to 1, indicating that the variance structure resemble that of a gamma distribution. The parameter estimates have not changed much compared with model fit that assumes constant variance.

The residual standard error is 1.126683 on 208 degree of freedom has also improved as against 23.57. Logarithm of the likelihood function is evaluated to be -940.5496.

However, comparing models A and B does make sense as they are both nested models. The log-likelihood ratio test and p-value are indicated in the table above.

The fitted Exponential regression curve is given by the following expression:

$$\hat{Y}_i = 3.834931 \exp\left(x_i / 23.301546\right) \text{Yi}$$

Table 3: The Summary Output of F-Test.

Model	df	AIC	BIC	logLik	test	L.Ratio	p-value
A	3	1927.09	1937.14	-960.31			
B	4	1888.43	1901.822	-940.21	1 vs 2	40.66609	0.0001

CONCLUSION

This study focused on the performance and estimation of nonlinear regression models with variance homogeneity and heterogeneity (a situation where the assumption that the residuals are normally distributed with mean 0 and variance σ is wrong) using Carbon monoxide dataset. The residual standard errors attached to the two models in (Tables 1 and 2), the log likelihood ratio test and the p-value in (Table 3) indicate that the model with different variances is better.

SELF-STARTER FUNCTIONS

In this study, we make use of nlrwr and nlme packages in R. The following code were used for the data analysis.

The constructed self-starter functions for exponential model are:

```
expModelInit <- function(mCall,
LHS, data) {
xy <- sortedXyData(mCall[["predictor"]],
LHS, data)
mFit <- lm(log(xy[, "y"]) ~
xy[, "x"])
coefs <- coef(lmFit)
alpha <- exp(coefs[1])
beta <- 1/coefs[2]
value <- c(beta, alpha)
names(value) <- mCall[c("beta",
"alpha")]
value
   }
SSexp <- selfStart(expModel, expModelInit,
c("beta", "alpha"))
getInitial(co ~ SSexp(Traffic density,beta,
alpha), data = mod)
mod.m1 <- nls(co ~ SSexp(Traffic density,
beta, alpha), data = mod)
summary(mod.m1)
 mod.m2 <- gnls(co ~ SSexp(Traffic.density,
+ beta,alpha), data = mod, weights =
varPower())
> summary(mod.m2)
```

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