

A Concrete Adomian Decomposition Method for Quadratic Riccati's Differential Equations.

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ABSTRACT

In this paper, Adomian decomposition method is applied to the general Riccati's differential equations. Two test problems were used to validate the reliability of the method and the results showed that Adomian decomposition method is in agreement with the analytic or the classical method of solving quadratic Riccati differential equations. The large scale computing and symbolic representation was facilitated by the Maple 14 software package.

(Keywords: Adomian decomposition method, quadratic Riccati differential equations)

INTRODUCTION

Riccati Differential Equations (RDE) are named after the Italian Nobleman Jacopo Francesco Riccati. RDEs have been widely studied [5] and [7] and they have equally wide application in random processes, optimal control, and diffusion problems. Beside the important engineering and scientific applications, RDE are widely applicable in stochastic theory and the network synthesis. The solution to this type of equations can be found by classical and numerical method such as forward Euler and Runge-Kutta method.

Very recently, [8] employed the analytic technique called Homotopy analysis to solve RDE. Variation iteration method has also been used by [1] to solve RDE with constant coefficient. Differential transform method is also a powerful mathematical tool for approximating linear and nonlinear differential equations. For nonlinear differential equations, an "unconditional stable scheme" by Adomian decomposition method (ADM) has been presented by [2], [9]. The ADM is widely used to obtain exact and numerical solutions to nonlinear differential equations [4], [3]. The convergence of

this powerful method has been given by [6]. In this paper we apply it to the quadratic Riccati differential equations.

THE ADOMIAN DECOMPOSITION METHOD FOR QUADRATIC RDE

The general form of RDE is given as:

$$y' = p(t) + q(t)y + r(t)y^2 \quad (1)$$

By ADM, Equation (1) can be given in linear and nonlinear operator form as:

$$Ly = f + Ry + Ny \quad (2)$$

where $Ly = y'$, $Ry = p(t) + q(t)y$, $Ny = r(t)y^2$ and $f = 0$. Applying inverse linear operator on equation 2, we obtain:

$$y(t) = g(t) + L^{-1}Ry + L^{-1}Ny \quad (3)$$

where $g(t)$ is the initial/boundary condition and L^{-1} is a single integral in RDE.

Equation (3), by ADM is given as:

$$\sum_{n=0}^{\infty} y_n(t) = y_0 + L^{-1}[p(t)] + \mathfrak{I} \quad (4)$$

where

$$L^{-1}q(t) \sum_{n=0}^{\infty} y_n(t) + L^{-1} \sum_{n=0}^{\infty} \lambda^n A_n$$

where λ

is a grouping parameter and A_n is the Adomian polynomial given as:

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} N(y) \Big|_{\lambda=0} \quad (5)$$

The first 10 Adomian polynomials of Equation (1) are as follows:

$$\begin{aligned} A_0 &= y_0^2 \\ A_1 &= 2y_0y_1 \\ A_2 &= 2y_0y_2 + y_1^2 \\ A_3 &= 2y_0y_3 + 2y_1y_2 \\ A_4 &= 2y_0y_4 + 2y_1y_3 + y_2^2 \\ A_5 &= 2y_0y_5 + 2y_1y_4 + 2y_2y_3 \\ A_6 &= 2y_0y_6 + 2y_1y_5 + 2y_2y_4 + y_3^2 \\ A_7 &= 2y_0y_7 + 2y_1y_6 + 2y_2y_5 + 2y_3y_4 \\ A_8 &= 2y_0y_8 + 2y_1y_6 + 2y_2y_5 + 2y_3y_5 \\ &\quad + y_4^2 \\ A_9 &= 2y_0y_9 + 2y_1y_8 + 2y_2y_7 + 2y_3y_6 \\ &\quad + 2y_4y_5 \\ A_{10} &= 2y_0y_{10} + 2y_1y_9 + 2y_2y_8 + 2y_3y_7 \\ &\quad + 2y_4y_6 + y_5^2 \end{aligned}$$

ADM gives the solution of equation (1) by the recursive relation:

$$\begin{aligned} y_0 &= g(t) \\ y_{n+1} &= L^{-1}Ry_n + L^{-1}A_n \end{aligned} \quad (6)$$

APPLICATION OF ADM TO QUADRATIC RDE

In this section, we discuss how to solve RDE numerically by ADM using Equations (1) – (6).

Problem 1

Consider the RDE,

$$\frac{dy}{dt} = 1 - y^2 \quad y(0) = 0 \quad (7)$$

The exact solution is given as:

$$y = \frac{e^{2t} - 1}{e^{2t} + 1} \quad (8)$$

The Taylor's series expansion of $y(t)$ in Equation (8) about $t = 0$ is given as:

$$y = t - \frac{t^3}{3} + \frac{2y^5}{15} - \frac{17t^7}{315} + \dots \quad (9)$$

Applying Equations (2) – (6) to Equation (7), we have:

$$\begin{aligned} y_0 &= t \\ y_1 &= -\int_0^t y_0^2 dt = -\frac{1}{3}t^3 \\ y_2 &= -2\int_0^t y_0y_1 dt = \frac{2}{15}t^5 \\ y_3 &= -\int_0^t (2y_0y_2 + y_1^2) dt = -\frac{17}{315}t^7 \\ y_4 &= -2\int_0^t (y_0y_3 + y_1y_2) dt = \frac{62}{2835}t^9 \\ y_5 &= -\int_0^t (2y_0y_4 + 2y_1y_3 + y_2^2) dt \\ &= -\frac{1328}{155925}t^{11} \\ y_6 &= -2\int_0^t (y_0y_5 + 2y_1y_4 + y_2y_3) dt \\ &= \frac{21844}{6081875}t^{13} \\ y_7 &= -2\int_0^t (y_0y_6 + y_1y_5 + y_2y_4 + \frac{1}{2}y_3^2) dt \\ &= -\frac{929569}{638512875}t^{15} \end{aligned}$$

$$y_8 = -2 \int_0^t (y_0 y_7 + y_1 y_6 + y_2 y_5 + y_3 y_4) dt$$

$$= \frac{6404582}{10854718875} t^{15}$$

$$y_9 = -2 \int_0^t (y_0 y_8 + y_1 y_7 + y_2 y_6 + y_3 y_5 + \frac{1}{2} y_4^2) dt = -\frac{443861162}{1856156927625} t^{19}$$

$$y_{10} = -2 \int_0^t (y_0 y_9 + y_1 y_8 + y_2 y_7 + y_3 y_6 + y_4 y_5) dt = \frac{18888466084}{194896477400625} t^{21}$$

Continuing in this order and taking a finite term of Equation (4), we have:

$$\sum_{n=0}^{12} y_n = t - \frac{1}{3} t^3 + \frac{2}{15} t^5 - \frac{17}{315} t^7 + \frac{62}{2835} t^9 - \frac{1382}{155925} t^{11} + \frac{21844}{6081075} t^{13} - \frac{929569}{638512875} t^{15} + -\frac{6404582}{10854718875} t^{17} \dots$$

The difference between the exact solution, Equation (8) and $\sum_{n=0}^{12} y_n$ with the absolute error E_A is given in Table I.

The obvious similarity between the exact solution and ADM method solution is further shown in Figures 1 and 2.

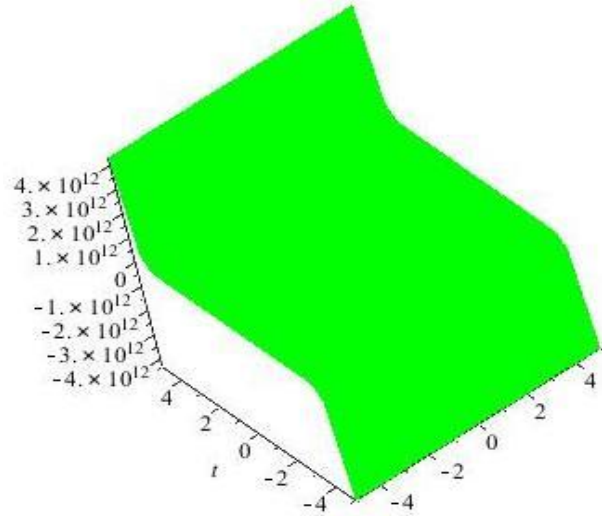


Figure 1: Solution of Problem 1 by ADM using $\sum_{n=0}^{12} y_n$

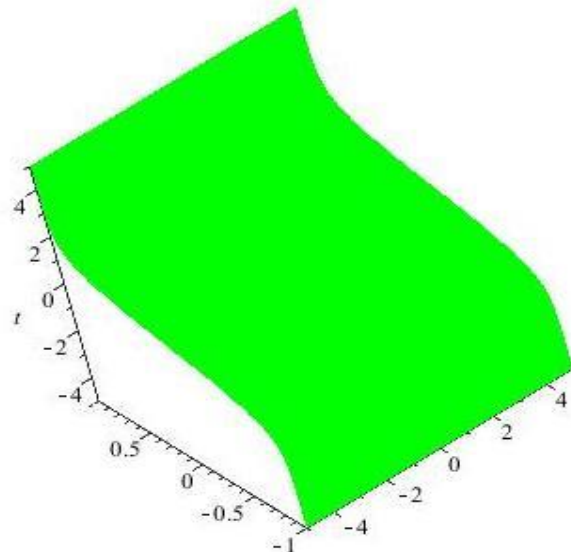


Figure 2: Exact Solution of Problem 1.

Table 1: Exact versus ADM Solution of Problem 1.

t	Exact Solution	Solution with ADM	E _A
0.1	0.09966799462495581711	0.09966799462495581711	0.00000000000000000000
0.2	0.19737532022490400073	0.19737532022490400074	0.00000000000000000000
0.3	0.29131261245159090582	0.29131261245159090586	0.00000000000000000000
0.4	0.37994896225522488527	0.37994896225522499442	0.00000000000000000000
0.5	0.46211715726000975851	0.46211715726005340501	0.00000000000004396483
0.6	0.53704956699803528586	0.53704956700379776707	0.00000000000576205750
0.7	0.60436777711716349631	0.60436777747087569932	0.0000000035371194862
0.8	0.66403677026784896369	0.66403678265426110508	0.0000001238641300638
0.9	0.71629787019902442081	0.71629815264237353687	0.0000028244334893124
1.0	0.76159415595576488812	0.76159874697497754088	0.0000459101921301830

Table 2: Exact versus ADM Solution of Problem 2.

t	Exact Solution	Solution with ADM	E _A
0.1	0.10000000000000000000	0.10000000000000000000	0.00000000000000000000
0.2	0.20000000000000000000	0.20000000000000000000	0.00000000000000000000
0.3	0.30000000000000000000	0.30000000000000000000	0.00000000000000000000
0.4	0.40000000000000000000	0.40000000000000000000	0.00000000000000000000
0.5	0.50000000000000000000	0.50000000000000000000	0.00000000000000000000
0.6	0.60000000000000000000	0.60000000000000000000	0.00000000000000000000
0.7	0.70000000000000000000	0.70000000000241917623	0.0000000000241906493
0.8	0.80000000000000000000	0.80000000015328562045	0.0000000015328494030
0.9	0.90000000000000000000	0.900000000465117145956	0.00000000465117100390
1.0	1.00000000000000000000	1.00000003743953667027	0.00000003743952992075

Problem 2

Consider the RDE

$$\frac{dy}{dt} = 1 + t^5 - 2t^4y + t^3y^2 \quad y(0) = 0 \quad (10)$$

The exact solution is given as $y = t$

Similarly, applying Equations (2) – (6) to Equation (10), we have:

$$y_0 = t + \frac{t^6}{6}$$

$$y_1 = \int_0^t (-2t^4y_0 + t^3y_0^2) dt$$

$$= -\frac{t^2}{6} + \frac{t^{16}}{576}$$

$$y_2 = \int_0^t (-2t^4y_1 + 2t^3y_0y_1) dt$$

$$= -\frac{t^{16}}{288} + \frac{t^{26}}{44928}$$

$$y_3 = \int_0^t (-2t^4y_2 + 2t^3y_0y_2 + t^3y_1^2) dt$$

$$= \frac{t^{16}}{576} - \frac{t^{26}}{14976} + \frac{5t^{36}}{17252357}$$

$$y_4 = \int_0^t (-2t^4y_3 + 2t^3y_0y_3 + 2t^3y_1y_2) dt$$

$$= \frac{t^{26}}{14976} - \frac{5t^{36}}{4313088} + \frac{t^{46}}{264536064}$$

Continuing in this order, we have:

$$\sum_{n=0}^{\infty} y_n = t - \frac{1}{3}t^3 + \frac{2}{15}t^5 - \frac{17}{315}t^7$$

$$+ \frac{62}{2835}t^9 - \frac{1382}{155925}t^{11} + \dots$$

The result of the exact solution and ADM solution of Problem 2 is shown in Table 2 and Figures 3 and 4.

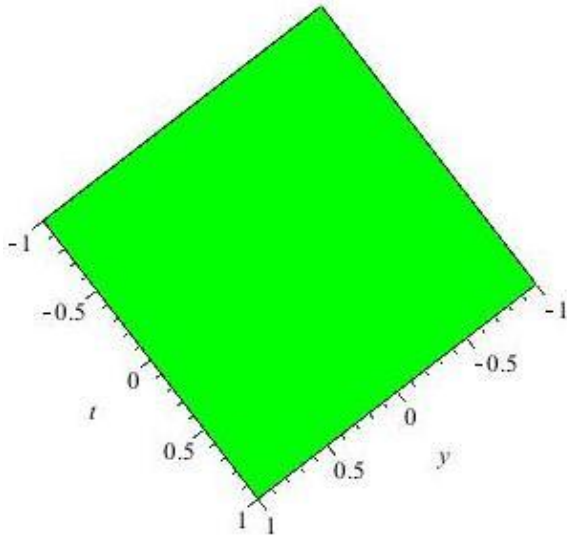


Figure 3: Exact Solution of Problem 2.

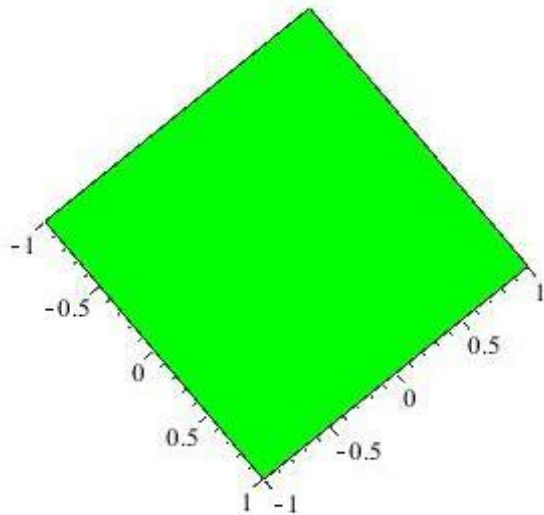


Figure 4: Solution of Problem 2 by ADM using

$$\sum_{n=0}^{10} y_n$$

Tables 1 and 2 show the efficiency of the result obtained by ADM with the exact solutions of RDE. This is further demonstrated in Figures 1-4.

From the comparison in the tables, we reported the Absolute error, E_A of the solution. Some

solution showed E_A equal to zero; which indicates that the results of ADM are in consonant with the exact solutions. Although, we evaluated finite terms of the decomposition series; Problem 1, thirteen terms and Problem 2, eleven terms. Higher order accuracy can be obtained by evaluating more components of Equation (4).

CONCLUSION

ADM is a very powerful mathematical tool for obtaining exact and numerical solutions to nonlinear ordinary and partial differential equations. In this paper, it has successfully been applied to obtain numerical and exact solution to RDEs. Although, finite terms of the Adomian series were considered in each test problem, the result to a large extent, when compared were about the same as the exact solutions. The tables and the plots showed the approximate and the exact result in each case. And they further confirmed the ability and reliability of ADM. Therefore, ADM can be widely applied in scientific fields where classical solutions to nonlinear ordinary differential equations are difficult to obtain.

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