On Adomian Decomposition Method (ADM) for Numerical Solution of Ordinary Differential Equations Arising from the Natural Laws of Growth and Decay.

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ABSTRACT

In this paper, a relatively new numerical method called Adomian Decomposition Method (ADM) for the numerical solution of ordinary differential equations arising from the natural laws of growth and decay shall be presented. We assume that growth and decay can be modeled into a simple first order differential equation of the form $y^{'} = f(x, y),\ y(x_0) = y_0$. Some numerical results are generated and comparisons are made between the numerical and theoretical results.

(Keywords: ADM, growth, decay, approximations and convergence)

INTRODUCTION

It has been observed that some of the existing methods for the numerical solution of the problems of the form,

$$y^{'} = f(x, y),\ y(x_0) = y_0,\ x \in\ a, b \qquad \textbf{(1)}$$

are based on the principle of discretization and they only allow the solution to a given initial value problem at a given interval. The above deficiency leads to a situation where some fundamental phenomena are easily avoided. In this paper, we present a relatively new numerical method namely Adomian Decomposition Method (ADM). It provides a good approximation to both linear and non-linear problems. The method is quantitative rather than qualitative. It is analytic throughout the positive region and it requires neither linearization nor perturbation. It is also continuous with no resort to discretization. The method provides the solution as an infinite series in which each term is determined.

Throughout, we shall consider an equation of the form (1) which occurs most often in biological and chemical sciences. Let $f(t)$ represents the amount of a quantity at a time t. If we assume that the rate of change of $f(t)$ is proportional to $f(t)$, then we have the following differential equation:

$$\frac{d}{dt} f(t) = \lambda f(t) \qquad \textbf{(2)}$$

where λ is a constant. If $f(t)$ decreases, then λ<0 and Equation (2) is called law of natural decay. If $f(t)$ increases, then λ>0 and (2) is called law of natural growth. We shall now proceed to discuss the basic concept of ADM.

THE BASIC CONCEPT OF ADOMIAN DECOMPOSTION METHOD (ADM)

The method consists of splitting the given equation into linear and non-linear parts, inverting the highest-order derivative operator contained in the linear operator on both sides, identifying the initial conditions and the terms involving the independent variables alone as initial approximation, decomposing the unknown function into a series, whose components can be easily computed, decomposing the non-linear function in terms of polynomials called "Adomian Polynomials" and finding the successive terms of the series solution by recurrent relation using the polynomials obtained (c.f. Adomian, 1998). The ADM can effectively handle higher order initial value problems (c.f. Wazwaz, 2000).

To solve the problem of the form (1), we write it in an operator form as:

$$Ly = f(x, y) \qquad \textbf{(3)}$$

where the differential operator L is given by:

$$L = \frac{d}{dx} \qquad \textbf{(4)}$$

The inverse operator:

$$L^{-1} = \int_0^x dx \qquad \textbf{(5)}$$

If we operate L^{-1} on (3) and impose the initial condition $y(0) = y_0$, we obtain

$$y(x) = y_0 + L^{-1} \ f(x, y) \qquad \textbf{(6)}$$

The Adomian Decomposition Method assumes a series solution for $y(x)$ given by an infinite sum of components:

$$y(x) = \sum_{n=0}^{\infty} y_n(x) \qquad \textbf{(7)}$$

where the components $y_n(x)$ will be determined recursively. Moreover, the method defined the non-linear function $f(x, y)$ by the infinite series of polynomial,

$$f(x, y) = \sum_{n=0}^{\infty} A_n \qquad \textbf{(8)}$$

The so-called Adomian Polynomial A_n can be calculated for various classes of non-linearity according to algorithm formally set by Wazwaz (2000). If we now substitute (7) and (8) into (6), we obtain:

$$\sum_{n=0}^{\infty} y_n(x) = y_0 + L^{-1}\left[\sum_{n=0}^{\infty} A_n\right] \qquad \textbf{(9)}$$

The next step is to seek a way to determine the component $y_n(x)$ for which $n \geq 0$. We first identify the zeroth component $y_0(x)$ by the term which arises from the initial condition. The remaining components are determined by using the preceding components (Adomian, 1994). Each term of the series in (9) is given by the recurrent relations:

$$y_0(x) = y_0 \qquad \textbf{(10)}$$

$$y_{n+1}(x) = L^{-1}(A_n) \qquad \textbf{(11)}$$

We must state here that in practice all the terms of the series in (7) cannot be determined and the solution will be approximated by series of the form,

$$\phi_n(x) = \sum_{n=0}^{\infty} y_n(x) \qquad \textbf{(12)}$$

By using the method described above, we obtain series solution for Equation (1). The method reduces significantly the massive computation which may arise if discretization methods are used for the solution of non-linear problems.

APPLICATIONS AND NUMERICAL EXPERIMENTS

We shall now apply ADM to problems arising from the Laws of growth and decay and other related problems.

Problem 1

Let the growth of a certain micro organism be represented by the differential equation $\frac{dN}{dt} = \lambda N, N(0) = 1000.$ We assume that the model grows continuously and without restriction. One might ask, how many of those micro organisms exist if an individual produces an average of 0.2 offspring every hour?

The above problem can be solved analytically to give:

$$N(t) = 1000e^{0.2t} \qquad \textbf{(13)}$$

By using ADM, we have:

$$LN = \lambda N \qquad \textbf{(14)}$$

If we find the inverse of (1) and use the initial condition, we obtain:

$$N = 1000 + \lambda L^{-1}(N) \qquad \textbf{(15)}$$

$$N_0(t) = 1000$$

$$N_{n+1} = \lambda L^{-1}(N_n)$$

Hence,

$$N_1(t) = \int_0^t N_0 dt = \lambda(1000t)$$

$$N_2(t) = \int_0^t N_1 dt = \frac{\lambda^2(1000t^2)}{2!}$$

$$N_3(t) = \int_0^t N_2 dt = \frac{\lambda^3(1000t^3)}{3!}$$

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$$N_n(t) = \int_0^t N_{n-1} dt = \frac{\lambda^n(1000t^n)}{n!}$$

The details of the computed result are shown in Table 1 below.

Problem 2

A certain radioactive substance is known to decay at the rate proportional to the amount present. A block of this substance having a mass of $100g$ originally is observed. After 40 hours, its mass reduces to $90g$. Find an expression for the mass of the substance at any time and apply the ADM to solve this problem for $t \in \ 0,1$.

The above problem has a differential equation of the form;

$$\frac{dN}{dt} = -\lambda N,\ N(0) = 100,\ t \in \ 0,1 \qquad \textbf{(16)}$$

where N represents the mass of the substance at any time t and λ is a constant which specifies the rate at which these particular substance decays. Thus the theoretical solution to equation (16) is given by:

$$N(t) = 100e^{-0.0026t} \qquad \textbf{(17)}$$

Table 1: Performance of ADM on $\frac{dN}{dt} = \lambda N, N(0) = 1000, t \in \ 0,1$

t	ADM	EXACT SOLUTION	ERROR
0.00	1000.000000	1000.000000	0.000000
0.10	1020.201340	1020.201340	0.000000
0.20	1040.801774	1040.801774	0.000000
0.30	1061.836571	1061.836547	0.000024
0.40	1083.287068	1083.287068	0.000000
0.50	1105.170930	1105.170918	0.000012
0.60	1127.496852	1127.496852	0.000000
0.70	1150.273799	1150.273799	0.000000
0.80	1173.510895	1173.510871	0.000024
0.90	1197.217363	1197.217363	0.000000
1.00	1221.402758	1221.402758	0.000000

Hence, (17) is the expression for the mass of the substance at any time t. We now apply ADM to (16):

$$LN = -\lambda N \quad \textbf{(18)}$$

If we find the inverse of (18) and use the initial condition, we obtain:

$$N = 100 - \lambda L^{-1}(N) \quad \textbf{(19)}$$

$$N_0(t) = 100$$

$$N_{n+1} = -\lambda L^{-1}(N_n)$$

Hence,

$$N_1(t) = \int_0^t N_0 dt = -\lambda(100t)$$

$$N_2(t) = \int_0^t N_1 dt = \frac{(-\lambda)^2(100t^2)}{2!}$$

$$N_3(t) = \int_0^t N_2 dt = \frac{(-\lambda)^3(100t^3)}{3!}$$

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$$N_n(t) = \int_0^t N_{n-1} dt = \frac{(-\lambda)^n(100t^n)}{n!}$$

The detailed result is shown in table 2 below.

Problem 3

Consider the initial value problem

$$y' = y,\ y(0) = 1,\ x \in 0,1 \quad \textbf{(20)}$$

with the theoretical solution $y(x) = e^x$.

Applying ADM operator to (20) produces

$$Ly = y \quad \textbf{(21)}$$

We now operate L^{-1} on both sides of (21) and use the initial condition, we obtain:

$$y(x) = y(0) + L^{-1}(y) = 1 + L^{-1}(y)$$

$$\sum_{n=0}^{\infty} y_n(x) = 1 + \int_0^x \sum_{n=0}^{\infty} y_n \quad (22)$$

$$y_0(x) = 1$$

$$y_{n+1} = L^{-1}(y_n),\ n \geq 0$$

Table 2: Performance of ADM on $\frac{dN}{dt} = -\lambda N,\ N(0) = 100,\ t \in 0,1$

t	ADM	EXACT SOLUTION	ERROR
0.00	100.00000000	100.00000000	0.00000000
0.10	99.97400340	99.97400338	0.00000002
0.20	99.94801353	99.94801352	0.00000001
0.30	99.92203041	99.92203041	0.00000000
0.40	99.89960541	99.89960541	0.00000000
0.50	99.87008449	99.87008446	0.00000003
0.60	99.84412162	99.84412162	0.00000000
0.70	99.81816555	99.81816552	0.00000003
0.80	99.79221618	99.79221617	0.00000001
0.90	99.76627357	99.76627357	0.00000000
1.00	99.74033771	99.74033771	0.00000000

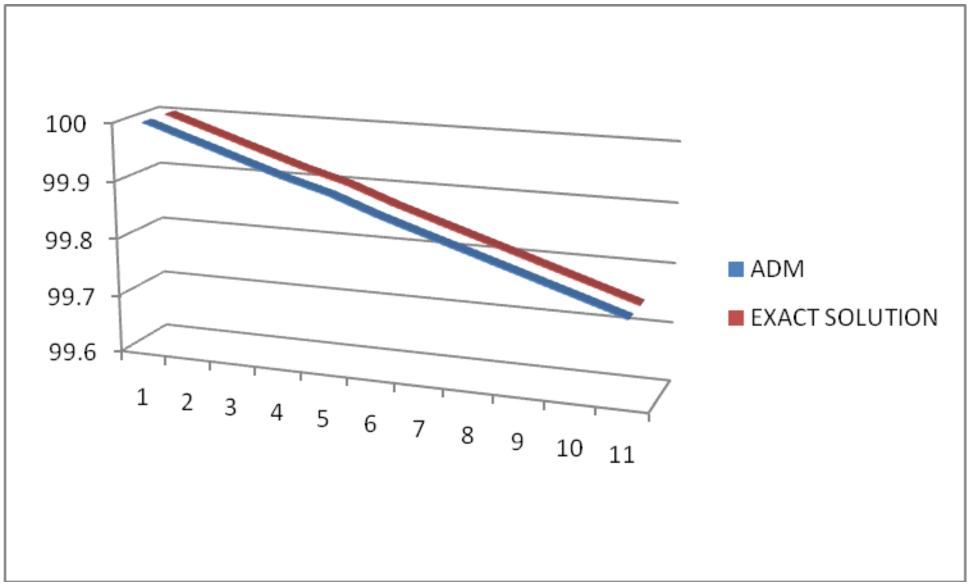

Figure 1: Graphical Representation of ADM for Problem 2.

We can then proceed to compute the first few terms of the series:

$$y_1(x) = \int_0^x y_0 dx = x$$

$$y_2(x) = \int_0^x y_1 dx = \frac{x^2}{2!}$$

$$y_3(x) = \int_0^x y_2 dx = \frac{x^3}{3!}$$

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$$y_n(x) = \int_0^x y_{n-1} dx = \frac{x^n}{n!}$$

Thus,

$$\phi_{10}(x) = \sum_{n=0}^{9} y_n(x),\ n \geq 0 \quad \textbf{(23)}$$

For application purpose, only the first ten terms of the series are computed. Table 3 below presents the computed results.

Table 3: Performance of ADM on $y' = y,\ y(0) = 1,\ x \in [0,1]$

x	ADM	EXACT SOLUTION	ERROR
0.00	1.0000000	1.0000000	0.0000000
0.10	1.1051709	1.1051709	0.0000000
0.20	1.2214028	1.2214028	0.0000000
0.30	1.3498586	1.3498588	0.0000002
0.40	1.4918246	1.4918247	0.0000001
0.50	1.6487214	1.6487213	0.0000001
0.60	1.8221186	1.8221188	0.0000002
0.70	2.0137527	2.0137527	0.0000000
0.80	2.2255411	2.2255409	0.0000002
0.90	2.4596033	2.4596031	0.0000002
1.00	2.7182819	2.7182818	0.0000001

Problem 4

Consider the initial value problem:

$$y' = y^2, \; y(0) = 1 \qquad \textbf{(24)}$$

with theoretical solution $y(x) = \dfrac{1}{1-x}, \; 0 \le x < 1$

We apply ADM operator to (24) to produce:

$$Ly = y^2 \qquad \textbf{(25)}$$

If we find the inverse of (25) and use the initial condition, we obtain:

$$\sum_{n=0}^{\infty} y_n(x) = 1 + L^{-1} \sum_{n=0}^{\infty} A_n$$

$$y(x) = y(0) + L^{-1}(y^2) = 1 + L^{-1}(y^2)$$

With $y_0(x) = 1$

and

$$y_{n+1}(x) = L^{-1}(A_n), \; n \ge 0$$

Applying the same procedure as in the problems above, we obtain the computed results below.

Table 4: Performance of ADM on $y' = y^2, \; y(0) = 1, \; 0 \le x < 1$

x	ADM	EXACT SOLUTION	ERROR
0.00	1.000000000	1.000000000	0.000000000
0.10	1.111111164	1.111111164	0.000000000
0.20	1.249999881	1.250000000	0.000000119
0.30	1.428571224	1.428571463	0.000000238
0.40	1.666666746	1.666666627	0.000000119
0.50	2.000000000	2.000000000	0.000000000
0.60	2.500000000	2.500000000	0.000000000
0.70	3.333333969	3.333333969	0.000000000
0.80	5.000000000	5.000000000	0.000000000
0.90	10.000000476	10.000000000	0.000000476

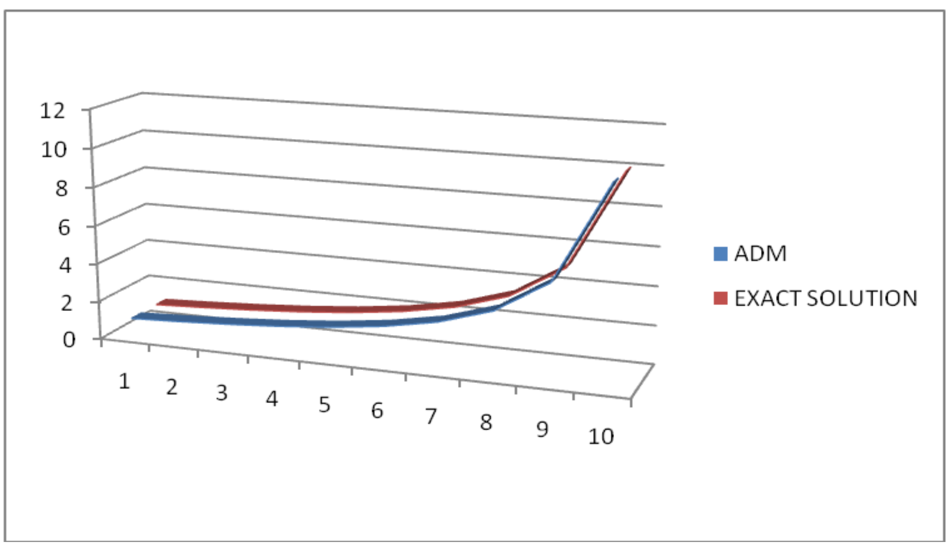

Figure 2: Graphical Representation of ADM for Problem 4.

CONCLUSION

We have been able to present and apply the ADM to first order differential equations. We have also presented and compared the numerical solutions of ADM with the theoretical (exact) solutions. From our findings, we observed that better accuracy can be obtained by accommodating more terms in our decomposition series. One of the advantages of ADM is that it generates the solutions over infinite intervals and that it also converges to the exact solutions.

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