On the Two-Parameter Bifurcation in a Predator-Prey System of Ivlev Type.

Prof. Maligie Surrah Sesay; Bala Ma'aji Abdulhamid, Ph.D.*; and Suleiman Onimisi Aliyu

Mathematical Sciences Programme, Abubakar Tafawa Balewa University,
PMB 0248, Bauchi, Nigeria.

*E-mail: bmabdulhamid67@yahoo.com

ABSTRACT

In this paper we consider conditions for the existence and non-existence of a critical point in the first quadrant for a predator-prey system of Ivlev type. A geometrical interpretation of an existing theorem is then presented and a condition, which is necessary and sufficient, established by Sugie (1998) for a unique stable limit cycle is obtained through linearization and utilization of properties of the Jacobian matrix. Examples are also given to illustrate our results.

(Keywords: predator prey system, Ivlev type, Jacobian matrix, critical point)

INTRODUCTION

We consider a predator-prey system of the form:

$$\begin{aligned} \dot{x} &= rx(1-x) - (1-e^{-ax})y \\ \dot{y} &= y((1-e^{-ax}) - D), \end{aligned} \qquad \textbf{(1)}$$

where x and y are the prey and predator population (or density), respectively; r, a, and D are positive parameters. This system is said to have the functional response $(1-e^{-ax})$ of the Ivlev type. There are quite a number of studies on the existence and uniqueness of limit cycle of Equation (1); see for example, Kuang and Freedman (1988), Koiji and Zegeling (1996), Sugie *et al*. (1997), Sugie (1998), and Attili (2001).

Attili and Mallak (2006) consider a predator-prey system with the functional response of the form $\varphi(x) = arctan(ax); a > 0$. The main concern is the existence of limit cycles for such system. A necessary and sufficient condition for the nonexistence of limit cycles is given for such a system.

In the study of predator-prey model with Ivlev functional response and impulsive perturbations, Baek (2007) proves that there exists a stable prey-free solution when the impulsive period is less than the critical value. Also, he finds a sufficient condition that the model is permanent.

In another study by Beak (2008) of a predator-prey system with Michaelis-Menten functional response and impulsive perturbations, which contains chemical and biological control terms, he applied the Floquet theory and establishes condition for the existence and stability of prey-free solutions of the system.

The main concern of this paper is a necessary and sufficient condition under which system (1) has exactly one limit cycle. Our work is based on Sugie (1998) who obtains (2) by transforming (1) to a Liénard system and utilizing its properties. The same condition (2) in the following Theorem will be obtained through linearization of system (1).

Theorem 1

If $$a > -\frac{2D+(1-D)\log(1-D)}{D-(1-D)\log(1-D)}\log(1-D), \qquad \textbf{(2)}$$

then the system (1) has a unique stable limit cycle; otherwise, system (1) has no limit cycle.

We shall deduce the same result from the eigenvalues and the properties of the Jacobian matrix of the linearized form of (1). The assumption:

$$D < 1 - e^{-a} \qquad \textbf{(3)}$$

in Sugie (1998) will also be presented from the linearized form and will be utilized in the discussion of the existence of limit cycles.

PRELIMINARY ANALYSIS OF PREDATOR-PREY SYSTEM

Taylor's expansion of e^{-ax} about $x = 0$, transforms the system (1) to form:

$$\dot{x} = rx - rx^2 - R_1(x,y)$$
$$\dot{y} = -Dy - R_2(x,y), \qquad \textbf{(4)}$$

where $R_1(x,y) = -rx^2 - \sum_{n=1}^{\infty} \frac{(-ax)^n}{n!} y$ and $R_2(x,y) = -\sum_{n=1}^{\infty} \frac{(-ax)^n}{n!} y$.

Equivalently, we have in matrix form, $\binom{\dot{x}}{\dot{y}} = A\binom{x}{y} + 0(2)$, where the matrix A is the linear part and $0(2)$ represents terms of R_1 and R_2 with degree 2 or higher in x and y.

Assuming that $detA \neq 0$, the eigenvalues of A, are of opposite signs since $r > 0$ and $D > 0$. The origin is a saddle point for the linear system

$$\binom{\dot{x}}{\dot{y}} = A\binom{x}{y}, \qquad \textbf{(5)}$$

and it is also a saddle point for the nonlinear system

$$\binom{\dot{x}}{\dot{y}} = A\binom{x}{y} + 0(2). \qquad \textbf{(6)}$$

The system (5) is called the linear approximation of (6).

LINEARIZATION OF PREDATOR-PREY SYSTEM

Next, we consider the critical point (λ, γ)in the first quadrant { (x,y): x>0, y>0 },

where $\lambda = -\frac{1}{a} log(1-D) \quad and \quad v = \frac{r}{D}\lambda(1-\lambda)$.

Define the new variables $\xi = x - \lambda \; and \; \eta = y - \gamma$, and at (ξ, η) the system (1.1) is transformed to

$$\binom{\dot{\zeta}}{\dot{\eta}} = \begin{pmatrix} r(1-2\lambda) - e^{-a\lambda} a\gamma & -D \\ (1-D)\,a\gamma & 0 \end{pmatrix} \binom{\zeta}{\eta} + 0(2), \qquad \textbf{(7)}$$

where $0(2)$ represents terms of degree 2 or higher in $\zeta \; and \; \eta$.

Let $\quad M = \begin{pmatrix} r(1-2\lambda) - e^{-a\lambda} a\gamma & -D \\ (1-D)a\gamma & 0 \end{pmatrix}, \qquad$ **(8)**

then the associated characteristic equation of the linear approximation of the system (7) is given by:

$$\sigma^2 - \sigma[r(1-2\lambda) - (1-d)a\gamma] + aD(1-D)\gamma = 0. \qquad \textbf{(9)}$$

The dynamics of the system (1) at (λ, γ) will be analyzed in below.

Let $P(x,y) = rx(1-x) - (1-e^{-ax})y$ and $Q(x,y) = y((1-e^{-ax}) - D)$, then the system (1) can be linearized more easily at $(\lambda, \gamma)$by computing the partial derivatives of P and Q at (λ, γ), that is, $P_x(\lambda,\gamma), P_y(\lambda,\gamma), Q_x(\lambda,\gamma) and\; Q_y(\lambda,\gamma)$. The matrix M obtained above is the same as the Jacobian matrix of the linearized form (1).

The eigenvalues of the matrix M are the roots of the characteristic equation $\sigma^2 - \tau\sigma + \delta = 0$, where τ and δ are the trace and determinant of M, respectively. We maintain the assumption that $\delta \neq 0$. Thus, the eigenvalues of M are $\sigma = \frac{\tau \pm \sqrt{\tau^2 - 4\delta}}{2}$, and the three cases for δ are discussed below.

EXISTENCE AND NON-EXISTENCE OF CRITICAL POINTS

The importance of maintaining the assumption that $\delta \neq 0$ has to do with the existence of critical point and the region in the xy-plane where it can possibly be found if it exists. When $\delta = a(1-D)r\lambda(1-\lambda) \neq 0$, this implies that $a(1-D)r\lambda\,(1-\lambda)$ is either less than zero or greater than zero and this leads to a contradiction of three cases.

<u>Case A</u>: We consider first the case foe which δ is negative, that is,

$$\delta = a(1-D)r\lambda(1-\lambda) < 0.$$

Since $\lambda = -\frac{1}{a} log(1-D)$, we have,

$$\delta = ar(1-D)(-\frac{1}{a}log(1-D))(1+\frac{1}{a}log(1-D)) < 0,$$

and

$$\delta = -\frac{1}{a}\,log(1-D)(1+\frac{1}{a}log(1-D)) < 0,$$

noting that $a > 0, r > 0, and\ (1 - D) > 0$.

Further simplification gives $D > 1 - e^{-a}$ if $\lambda > 1$. This means that the assumption $D < 1 - e^{-a}$ in Sugie (1998) fails. So the system (1) has no critical point in the first quadrant and therefore no limit cycle of the system exits, according to him.

Case B: If $\delta = a(1 - D)r\lambda(1 - \lambda) > 0$, then $\lambda(1 - \lambda) > 0$, since $a > 0, r > 0$, and $D < 1$. But $\lambda < 1$ implies that $(1 + \frac{1}{a} log(1 - D)) > 0$. The inequality further implies that $D < 1 - e^{-a}$ if $\lambda < 1$. Assumption (2) in Sugie (1998) holds.

Case C: If $\lambda = 1$, then $1 = -\frac{1}{a} log(1 - D)$ and $\gamma = 0$. This case implies that $\delta = 0$ and the critical point is not in the first quadrant but on the x-axis. This analysis adds a dimension to the two-parameter bifurcation in a predator-prey system of Ivlev type. It is clear from this analysis that the non-existence of critical points is linked to the condition that $\lambda > 1 and\ D > 1 - e^{-a}$. The existence of critical point is related to $\lambda < 1$ and $D < 1 - e^{-a}$. The two cases (A and B) above will make useful and meaningful contribution to the existence and non-existence of limit cycle when it is given a geometrical interpretation and additional conditions are assumed for the parameter A.

EXISTENCE AND NON-EXISTENCE OF LIMIT CYCLES

Three cases of the parameter a are implied by the cases above, namely;

$a < -log(1 - D)$,
$a > -log(1 - D)$,
and $a = -log(1 - D)$.

The third of the three cases gives the relationship between a and D as $D = 1 - e^{-a}$, which is crucial in giving the following Lemma a geometrical interpretation later.

Lemma 1. System (1) has at most one limit cycle if $a > 2$, if it exists it is hyperbolic.

Proof. If $a > 2$, then $1 - e^{-a} > 1 - e^{-2}$ since $D = 1 - e^{-a}$ is strictly increasing on $(2, \infty)$. If in addition $\lambda < 1$, then we have $D < 1 - e^{-a}$ and it follows that $D < 1 - e^{-2}$. Since $D < 1 - e^{-2}$ holds whenever $D < 1 - e^{-a}$ holds, it means that there is a limit cycle if $a > 2$.

Lemma 2. If $0 < a \leq 2$, then the system (1) has no limit cycles.

Proof. Suppose that $0 < a \leq 2$, then $0 < 1 - e^{-a} \leq 1 - e^{-2}$. This means that the curve, $D = 1 - e^{-a}$, is increasing on the interval $(0, 2]$. If in addition $\lambda > 1$, then $D > 1 - e^{-a}$ implying that the region of interest is above the curve $D = 1 - e^{-a}$. Hence there is no critical point and it follows that no limit cycle exists.

The two lemmas above lead logically to the following theorem in Koiji and Zegleng (1996).

Theorem 1. System (1) has at most one limit cycle if $a > 2$; if it exists it is hyperbolic. If $0 < a \leq 2$, then the system has no limit cycles.

Proof. We only need to apply Lemma 1 to show the existence of a limit cycle if $a > 2$ and apply Lemma 2 to show the non-existence of a limit cycles if $0 < a \leq 2$.

In order to achieve this, consider Figure 1 in which the line a=2 intersects the curve $D = 1 - e^{-a}$ in the aD-plane. The region between the curve $D = 1 - e^{-a}$ and the infinite strip $a > 2$ satisfies Lemma 1. The system (1) therefore has at most one limit cycle since $a > 2$ and it is hyperbolic. In the case of Lemma 2, consider the region above the curve $D = 1 - e^{-a}$ and the strip $0 < a \leq 2$. Therefore by Lemma 2, the system (1) has no limit cycle.

Remark: The two parameter bifurcation of (a, D) plane shown above is a direct consequence of the separation of the quadrant into two regions by the line $x = 1$.

This bifurcation line passes through (1,0) on the x-axis. Theorem 1 is therefore another consequence of this feature.

STABILITY IN PREDATOR-PREY SYSTEM

In this section we consider the characteristic equation:

$$\sigma^2 - \tau\sigma + \delta = 0 \qquad \textbf{(8)}$$

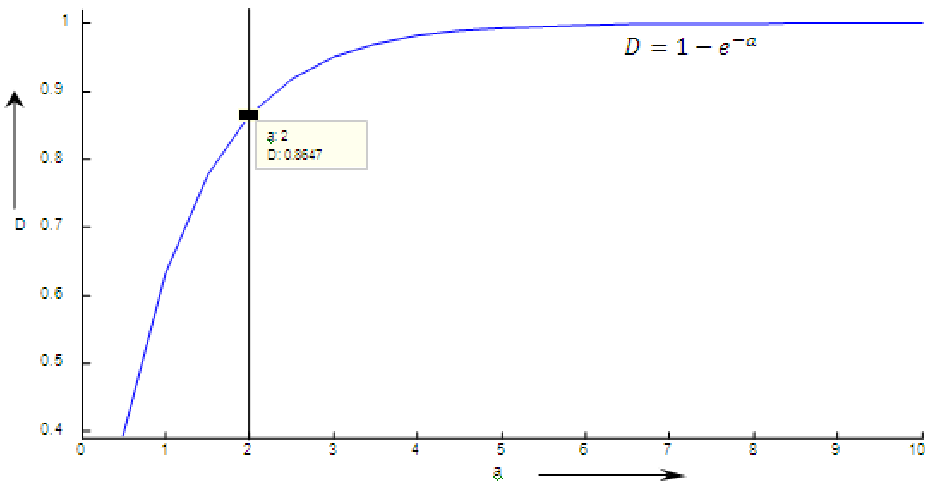

Figure 1: Two-parameter Bifurcation of the (a, D) showing 2-Regions: i) Region Above the Curve $D = 1 - e^{-a}$ and the Line $a = 2$ as a Region for the Non-Existence of Limit Cycles and ii) that Below the Curve and the Infinite Strip $a > 2$ as the Region for the Existence of Limit Cycle.

where τ is the trace and δ is the determinant of the Jacobian matrix M. The trace is given by $\tau = [r(1-2\lambda) - (1-D)a\gamma]$ and the determinant by $\delta = are^{-a\lambda}\lambda(1-\lambda)$.

The characteristic roots of Equation (8) are $\sigma = \frac{\tau \pm \sqrt{\tau^2 - 4\delta}}{2}$ and the discriminant is $\tau^2 - 4\delta$.

The analysis for stability and non-stability of the critical point, when it exists now follows using the trace τ and the determinant δ of the Jacobian matrix. The cases related to the trace of the Jacobian matrix of the linearized form of (1) are stated here as theorems using eigenvalues of the Jacobian matrix of M.

Theorem 2 (Case A). If the eigenvalues of the Jacobian matrix of the linearized form of (1) have opposite signs, then (1) has no critical point and there is no limit cycle.

Proof. Consider the characteristic equation $\sigma^2 - \tau\sigma + \delta = 0$ and for this case we have $\tau^2 - 4\delta > 0$ and $\delta < 0$. Since $\sigma = \frac{\tau \pm \sqrt{\tau^2 - 4\delta}}{2}$, the product of the two values of δ is negative, that is, $\sigma_1\sigma_2 = \delta < 0$. But $\delta = a(1-D)r\lambda(1-\lambda) < 0$. Substituting the value of λ and simplifying will yield $D > 1 - e^{-a}$. This inequality implies that there is no critical point and hence no limit cycle exists for the system (1).

Theorem 3 (Case B). If the eigenvalues of the Jacobian matrix of the linearized form of the system (1) are both positive, then there is critical point and a unique stable limit cycle.

Proof. Using the characteristic equation $\sigma^2 - \tau\sigma + \delta = 0$, its roots are $\sigma = \frac{\tau \pm \sqrt{\tau^2 - 4\delta}}{2}$. In this case $\tau^2 - 4\delta > 0$, $\tau > 0$ and $\delta > 0$. The second inequality implies that:

$$\tau = r + \frac{2r}{a} log(1-D) - \frac{a(1-D)r\lambda(1-\lambda)}{D} > 0, \quad \textbf{(9)}$$

and the third,

$$\delta = a(1-D)r\lambda(1-\lambda) > 0. \quad \textbf{(10)}$$

Substituting $\lambda = -\frac{1}{a} log(1-D)$ into (9) and simplifying will give:

$$a > -\frac{2D + (1-D) log(1-D)}{D + (1-D) log(1-D)} log(1-D) \quad \textbf{(11)}$$

Inequality (10) when simplified further yields $D < 1 - e^{-a}$ and it follows that there is a critical point in the first quadrant where $x(= \lambda) > 1$.

The critical point is unstable because the eigenvalues have positive real parts. Inequality (11) deduced from (9) is the necessary and sufficient condition in Sugie (1998); it follows by Theorem (1) that the system (1) has a unique stable limit cycle. This analysis also relates the stability of the critical point to the existence of a limit cycle.

The next case may be considered as a corollary to the preceding Theorem 3. It gives a condition for the non-existence or absence of limit cycles.

Corollary 1 (case C). If both eigenvalues of the Jacobian matrix have negative real parts, then the system (1) has no limit cycles.

Proof. This case involves three inequalities, $\tau^2 - 4\delta > 0$, $\tau < 0$ and $\delta > 0$. By the third inequality, $\delta > 0$, a critical point exists because $\lambda > 1 \; and \; D > 1 - e^{-a}$; it is stable since the eigenvalues have negative real parts. The second inequality $\tau < 0$ implies that,

$$\tau = r + \frac{2r}{a} log(1-D) - \frac{a(1-D)r\lambda(1-\lambda)}{D} < 0,$$

which on simplification, and using $\lambda = -\frac{1}{a} log(1-D)$, gives

$$a < -\frac{2D + (1-D)log(1-D)}{D + (1-D)log(1-D)} log(1-D). \qquad \textbf{(12)}$$

By Theorem 1, there are no limit cycles for this case.

EXAMPLES

Next we present examples to illustrate cases we have considered so far.

Example 1. Consider the system:

$$\begin{aligned} \dot{x} &= \tfrac{1}{2}x(1-x) - (1-e^{-\frac{1}{2}x})y \\ \dot{y} &= y((1-e^{-\frac{1}{2}x}) - \tfrac{6}{10}). \end{aligned} \qquad \textbf{(13)}$$

We relate the parameters in Equation (1) to (13) as follows,

$$r = \tfrac{1}{2}, D = 0.6, a = \tfrac{1}{2}, \; \lambda = 1.88 > 1.$$

Since $D > 1 - e^{-0.5}$, there is no critical point for Equation (13). The system does not have a limit cycle. In order to show this we compute the trace τ and the determinant δ of the Jacobian matrix for the system (13).

For the system the trace $\tau = -0.09 < 1$ and the determinant $\delta = -0.16 < 0$. Since $\tau < 0$, $\delta < 0$, and $\tau^2 - 4\delta = 0.65 > 0$, we apply Theorem 2 to conclude that system (13) does not have limit cycle.

Example 2. Consider the system:

$$\begin{aligned} \dot{x} &= \tfrac{1}{2}x(1-x) - (1-e^{-4x})y \\ \dot{y} &= y((1-e^{-4x}) - 0.5), \end{aligned} \qquad \textbf{(14)}$$

We relate the parameters in Equation (1) to (14) as follows,

$$r = \tfrac{1}{2}, D = 0.5, a = 4, \lambda = 0.17 < 1.$$

The computation of $\lambda = 0.17 < 1$ leads to the observation that a critical point exists and it is unstable for the system (14). Also trace $\tau = 0.05 > 1$ and the determinant $\delta = 0.14 > 0$. Since $\tau > 0$ and $\delta > 0$, we apply Theorem 2 to conclude that for system (14) there exists a unique stable limit cycle.

Example 3. Consider the system:

$$\begin{aligned} \dot{x} &= 2x(1-x) - (1-e^{-\frac{1}{4}x})y \\ \dot{y} &= y((1-e^{-\frac{1}{4}x}) - 0.5), \end{aligned} \qquad \textbf{(15)}$$

We compare the parameters in Equation (1) and (15) as follows,

$$r = 2, D = 0.5, a = \tfrac{1}{4}, \lambda = 0.88 > 0.$$

We observe that $\lambda = 0.88 < 1$ and $\delta = 0.528 > 0$. The value of $\tau = -1.56 < 0$ leads to the conclusion that there is no limit cycle for system (15) by Corollary 1.

Remark: This corollary is also saying that a critical point exists because $\lambda = 0.88 < 1$ and $D = 0.5 < 1 - e^{-0.25}$. This means that the non-existence of a limit cycle does not necessarily imply that a critical point does not exist. If a critical point exists, it is potentially a point contained in a limit cycle when the limit cycle exists.

CONCLUSION

In conclusion we consider two examples which give rise to the two questions posed by Sugie (1998):

i) Are there cases where system (1) has no limit cycles even if $a > 2$.

ii) What kind of condition is necessary for the system to have exactly one limit cycle?

We tend to agree with Sugie (1998) that Theorem 1 provides the answer to question (ii).

In the case of the question (i), the region $D > 1 - e^{-a}$ above the curve $D = 1 - e^{-a}$ and left of the line $a = 2$ provides the answer to the question. For example, choose the parameters in this region such that:

$$a = 3, D = 0.96 > 1 - e^{-3} = 0.95.$$

The value of λ satisfying this choice is $\lambda = 1.07 > 1$. A system with such parameters has no limit cycles even when $a > 2$.

REFERENCES

1. Attili Basem, S. and Mallak Saed, F. 2006. "Existence of Limit Cycles in a Predator-Prey System with a Functional Response of the Form Arctan(ax)". *Communications in Mathematical Analysis*. 1(1):33-40.

2. Attili Basem, S. 2001. "Existence of Limit Cycles in a Predator-Prey System with a Functional Response". *Int. J. Math. Sci.* 27:377-385.

3. Baek Hunki. 2007. "Complex Dynamics of a Ivlev-Type Predator-Prey System with Impulsive Control Strategy". Korean Society for Industrial and Applied Mathematics 2007 Annual Conference (KSIAM). Sungwoo Resort, Hoengseong, ROK. Nov. 23-24, 2007.

4. Baek Hunki. 2008. "Dynamics of a Predator-Prey System with Impulsive Control Strategy". Korean Society for Industrial and Applied Mathematics 2008 Annual Conference (KSIAM), POSTECH. Pohang, ROK. May 30-31, 2008.

5. Kooij, R.E. and Zegeling, A. 1996. "A Predator-Prey Model with Ivlev's Functional Response. *Journal of Mathematical Analysis and Application*. 198:473-489.

6. Kuang, Y. and Freedman, H.I. 1988. "Uniqueness of Limit Cycles in Gause Type Models of Predator-Prey Systems". *Math. Biosci*. 88:67-84.

7. Sugie, J., Kohno, R., and Miyazaki, R. 1997. "On a Predator-Prey System of Holling Type". *Proceedings of the A.M.S.* 125(7):2041-2050.

8. Sugie, J. 1998. "Two-Parameter Bifurcation in a Predator-Prey System of Ivlev Type". *J. of Mathematical Analysis and Application*. 217:349-371.

SUGGESTED CITATION

Sesay, M.S., B.M. Abdulhamid, and S.O. Aliyu. 2010. "On the Two-Parameter Bifurcation in a Predator-Prey System of Ivlev Type". *Pacific Journal of Science and Technology*. 11(1):266-271.

Pacific Journal of Science and Technology