On Some Moment Properties of Extreme Value Distributions with Applications to Daily Minimum Temperature Measurements.

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ABSTRACT

The choice of probability distribution for appropriate modeling of extreme-value measurements has been a challenging problem in the field of meteorology. This paper proposes the use of the Fisher information matrix method for choosing limiting minimum extreme-value distributions and modeling extreme-value temperature measurements. Using daily minimum temperature measurements, this paper also presents a comparative assessment of three extreme-value distributions namely; Frechet, Weibull, and Gumbel to obtain their parameters (location, scale and shape) by approximating the resulting information matrices to variance-covariance matrices using smith's proposition (1985).

(Keywords: meteorology, probability distribution, extreme-value measurements, extreme-value distributions, Frechet, Weibull, Gumbel, parameters)

INTRODUCTION

In the past, meteorologist have been faced with the problems of how to precisely handle extreme-value measurements of some quantities such as temperature, pressure, humidity, etc. However, modeling of such extreme-value measurements without error became very difficult. This is the core of this study.

As a matter of fact, temperature which measures the degree of coldness and hotness of a body, is of great importance to understanding nature. For instance, most biochemical reactions such as enzymatic reactions found in growth of plants and digestion in animals takes place only at certain temperature. It should be noted that minimum temperature measurements fall exactly to a class of extreme-value temperature measurements and hence follows extreme-value distributions (Fisher–Tippet Theorem).

The failure of meteorologists to precisely measure and model extreme-value measurements of this important element of weather and climate makes error-free weather prediction or forecasting impossible. This challenges underscores the need for this study.This paper considers Melbourne daily minimum temperature measurements for ten years (1995-2004) and presents a comparative analysis of each of these extreme-value distributions to estimate their scale, shape, and location parameters. The distribution with the least scale parameter estimate will best model the data.

MODEL SPECIFICATION AND METHOD

Name	Limit Maximum Distribution	Density Function
Frechet	$G_1(x_1\ \alpha) = \begin{cases} 1-\exp[-(-x)^{-\alpha}], & x \le 0, \alpha > 0 \\ 1, & x > 0 \end{cases}$	$g_1(x_1\ \alpha) = \begin{cases} -\alpha(-x)-\alpha-1\exp[-(-x)-\alpha], & x \le 0, \alpha > 0 \\ 0 & , x > 0 \end{cases}$
Weibull	$G_2(x_1\alpha) = \begin{cases} 1-\exp[-(-x)^{-\alpha}], & x \le 0, \alpha > 0 \\ 0 & , x \le 0, \alpha > 0 \end{cases}$	$g_2(x_1\alpha) = \begin{cases} \alpha x^{\alpha-1}\exp(-x^{\alpha}), & x > 0, \alpha > 0 \\ 0, & x < 0 \end{cases}$
Gumbel	$G_3(x) = \begin{cases} 1-\exp(x^{\alpha}), & -\infty < x < \infty \\ 0, & elsewhere \end{cases}$	$g_3(x) = \begin{cases} e^{x}e^{-ex}, & -\infty < x < \infty \\ 0, & elsewhere \end{cases}$

We are interested in the distribution that will best model our data.

The Fisher-Tippet Theorem

Let $\{x_n\}$ be a sequence of independent and identically distributed Random variables. If constant $a_n > 0, b_n \in \Re$ (a set of real numbers) and a non-degenerate distribution function $H \ni a_n^{-1}(m_n - b_n) \xrightarrow{d} H$. Then, H is the type of one of the following three distribution functions (i.e., the Frechet, the Weibull, and the Gumbel distribution).

It then follows that any extreme value measurement or data is taken to come from any of the three distributions. This validates the use of any of these distributions to model our data.

We are particularly interested in the distribution that will exactly model our data. In achieving this, the Fisher information matrix method and the Adeyemi and Ojo (2003) comparative method will be used. The diagonal element of the variance-covariance matrices gives the estimates of the location, scale and shape parameters respectively. The matrix with the least scale parameter estimate will best model the data.

The Information Matrix

The fisher information matrix is defined as:

$$I\ (\mu, \sigma, \alpha)\ \text{-E} \begin{pmatrix} \frac{\partial^2 InL}{\partial \mu^2} & \frac{\partial^2 InL}{\partial \mu \partial \sigma} & \frac{\partial^2 InL}{\partial \mu \partial \alpha} \\ \frac{\partial^2 InL}{\partial \sigma \partial \mu} & \frac{\partial^2 InL}{\partial \sigma^2} & \frac{\partial^2 InL}{\partial \sigma \partial \alpha} \\ \frac{\partial^2 InL}{\partial \alpha \partial \mu} & \frac{\partial^2 InL}{\partial \alpha \partial \sigma} & \frac{\partial^2 InL}{\partial \alpha^2} \end{pmatrix} \quad \textbf{(1)}$$

Where μ is the location parameter, σ the scale parameter, α the shape parameter and L the likelihood function. We obtain the Fisher information matrix for each of the distributions (as follows below).

INFORMATION MATRICES FOR THE DISTRIBUTIONS

We standardized the random variable x by introducing μ in conjunction with the location and scale parameter that is:

$$g_1(\mu, \sigma, \alpha) = \frac{\alpha}{\sigma}[-(\frac{y-\mu}{\sigma})^{-\alpha-1} \exp(-(\frac{y-\mu}{\sigma}))^{-\alpha}] \quad \textbf{(2)}$$

taking the likelihood, we have:

$$L_1 = \alpha^n \sigma^{-n} \pi_{i=1}^{n}[-(\frac{x-\mu}{\sigma}))^{-\alpha-1}[-\sum(\frac{x-\mu}{\sigma})^{-\alpha}] \quad \textbf{(3)}$$

taking the log likelihood, we have:

$$InL_1 = nIn\alpha - nIn\sigma - (\alpha+1)\sum In[-(\frac{x-\mu}{\sigma}) - \sum(-(\frac{x-\mu}{\sigma})^{-\alpha}] \quad \textbf{(4)}$$

taking the partial derivatives of the log likelihood, we obtain the information matrix for Frechet Distribution.

$$
I_1(y,\mu,\sigma,\alpha) = -E\begin{pmatrix}
\frac{(\alpha+1)}{\sigma^2}\sum(\frac{y-\mu}{\sigma})^{-2} - \alpha\frac{(\alpha+1)}{\sigma^2}\sum(\frac{y-\mu}{\sigma})^{-\alpha-2} & \frac{\alpha^2}{\sigma^2}\sum(\frac{y-\mu}{\sigma})^{-\alpha-1} & -\frac{1}{\sigma}\sum(\frac{y-\mu}{\sigma})^{-1} - \frac{\alpha}{\sigma}\sum(\frac{y-\mu}{\sigma})^{-\alpha-1}\ln(\frac{y-\mu}{\sigma}) + \frac{1}{\sigma}\sum(\frac{y-\mu}{\sigma})^{-\alpha-1} \\
\frac{\alpha^2}{\sigma^2}\sum(\frac{y-\mu}{\sigma})^{-\alpha-1} & \frac{-n\alpha}{\sigma^2} + \frac{\alpha(\alpha-1)}{\sigma^2}\sum(\frac{y-\mu}{\sigma})^{-\alpha} & \frac{n}{\sigma} + \frac{\alpha}{\sigma}\sum(\frac{y-\mu}{\sigma})^{-\alpha}\ln(\frac{y-\mu}{\sigma}) - \frac{1}{\sigma}\sum(\frac{y-\mu}{\sigma})^{-\alpha} \\
-\frac{1}{\sigma}\sum(\frac{y-\mu}{\sigma})^{-1} - \frac{\alpha}{\sigma}\sum(\frac{y-\mu}{\sigma})^{-\alpha-1}\ln(\frac{y-\mu}{\sigma}) + \frac{1}{\sigma}\sum(\frac{y-\mu}{\sigma})^{-\alpha-1} & \frac{n}{\sigma} + \frac{\alpha}{\sigma}\sum(\frac{y-\mu}{\sigma})^{-\alpha}\ln(\frac{y-\mu}{\sigma}) - \frac{1}{\sigma}\sum(\frac{y-\mu}{\sigma})^{-\alpha} & \frac{-n}{\sigma^2} - \sum(\frac{\alpha(\alpha-1)}{\sigma^2})^{-\alpha}[\ln(\frac{y-\mu}{\sigma})]^2
\end{pmatrix}
$$

(5)

Repeating the same steps for Weibull and Gumbel distribution, we have:

$$
I_2 = -E\begin{pmatrix}
\frac{(1-\alpha)}{\sigma^2}\sum(\frac{y-\mu}{\sigma})^{-2} + \frac{(1-\alpha)}{\sigma^2}\sum(\frac{y-\mu}{\sigma})^{\alpha-2} & \frac{\alpha^2}{\sigma^2}\sum(\frac{y-\mu}{\sigma})^{\alpha-1} & \frac{-1}{\sigma}\sum(\frac{y-\mu}{\sigma})^{-1} + \frac{\alpha}{\sigma}\sum(\frac{y-\mu}{\sigma})^{\alpha-1}\ln(\frac{y-\mu}{\sigma}) + \frac{1}{\sigma}\sum(\frac{y-\mu}{\sigma})^{\alpha-1} \\
\frac{-\alpha^2}{\sigma^2}\sum(\frac{y-\mu}{\sigma})^{\alpha-1} & \frac{n\alpha}{\sigma^2} + \frac{\alpha(\alpha-1)}{\sigma^2}\sum(\frac{y-\mu}{\sigma})^{\alpha} & \frac{-n}{\sigma} + \frac{1}{\sigma}\sum(\frac{y-\mu}{\sigma})^{\alpha}\ln(\frac{y-\mu}{\sigma}) + \frac{\alpha}{\sigma}\sum(\frac{y-\mu}{\sigma})^{\alpha} \\
\frac{-1}{\sigma}\sum(\frac{y-\mu}{\sigma})^{-1} + \frac{\alpha}{\sigma}\sum(\frac{y-\mu}{\sigma})^{\alpha-1}\ln(\frac{y-\mu}{\sigma}) + \frac{1}{\sigma}\sum(\frac{y-\mu}{\sigma})^{\alpha-1} & \frac{-n}{\sigma} + \frac{1}{\sigma}\sum(\frac{y-\mu}{\sigma})^{\alpha}\ln(\frac{y-\mu}{\sigma}) + \frac{\alpha}{\sigma}\sum(\frac{y-\mu}{\sigma})^{\alpha} & -\{\frac{-n}{\sigma} + [\ln(\frac{y-\mu}{\sigma})]^2 + \sum(\frac{y-\mu}{\sigma})^{\alpha}\}
\end{pmatrix}
$$

(6)

$$
I_2 = -E\begin{pmatrix}
\frac{(1-\alpha)}{\sigma^2}\sum(\frac{y-\mu}{\sigma})^{-2} + \frac{(1-\alpha)}{\sigma^2}\sum(\frac{y-\mu}{\sigma})^{\alpha-2} & \frac{\alpha^2}{\sigma^2}\sum(\frac{y-\mu}{\sigma})^{\alpha-1} & \frac{-1}{\sigma}\sum(\frac{y-\mu}{\sigma})^{-1} + \frac{\alpha}{\sigma}\sum(\frac{y-\mu}{\sigma})^{\alpha-1}\ln(\frac{y-\mu}{\sigma}) + \frac{1}{\sigma}\sum(\frac{y-\mu}{\sigma})^{\alpha-1} \\
\frac{-\alpha^2}{\sigma^2}\sum(\frac{y-\mu}{\sigma})^{\alpha-1} & \frac{n\alpha}{\sigma^2} + \frac{\alpha(\alpha-1)}{\sigma^2}\sum(\frac{y-\mu}{\sigma})^{\alpha} & \frac{-n}{\sigma} + \frac{1}{\sigma}\sum(\frac{y-\mu}{\sigma})^{\alpha}\ln(\frac{y-\mu}{\sigma}) + \frac{\alpha}{\sigma}\sum(\frac{y-\mu}{\sigma})^{\alpha} \\
\frac{-1}{\sigma}\sum(\frac{y-\mu}{\sigma})^{-1} + \frac{\alpha}{\sigma}\sum(\frac{y-\mu}{\sigma})^{\alpha-1}\ln(\frac{y-\mu}{\sigma}) + \frac{1}{\sigma}\sum(\frac{y-\mu}{\sigma})^{\alpha-1} & \frac{-n}{\sigma} + \frac{1}{\sigma}\sum(\frac{y-\mu}{\sigma})^{\alpha}\ln(\frac{y-\mu}{\sigma}) + \frac{\alpha}{\sigma}\sum(\frac{y-\mu}{\sigma})^{\alpha} & -\{\frac{-n}{\sigma} + [\ln(\frac{y-\mu}{\sigma})]^2 + \sum(\frac{y-\mu}{\sigma})^{\alpha}\}
\end{pmatrix}
$$

(7)

The respective information matrices for Weibull and Gumbel Distributions.

APPLICATION

The data will be fitted to each of the matrices. For convenience, we will assign values to our preliminary parameters that is, μ =0, σ =1, α =1, n=3650 so that we have:

$$
I_1 = \begin{pmatrix} -1,8275 & -0.6208 & 2.1987 \\ -0.6208 & 3650 & -3648.62 \\ -2.1987 & -3648.62 & 3642.12 \end{pmatrix}
$$

(8)

$$
I_2 = \begin{pmatrix} 0 & 3650 & -4341.65 \\ -0.6208 & 3650 & 2711.4 \\ -2.1987 & -3648.62 & 5675.12 \end{pmatrix}
$$

(9)

and

$$
I_3 = \begin{pmatrix} -8142367577000 & -22631912060000 \\ -2263192060000 & 628906053800000 \end{pmatrix}
$$

(10)

The inverse of these information matrices (Variance-Covariance Matrices) are respectively,

$$I_1^{-1} = \begin{pmatrix} 4.44 & 2.52 & -2.52 \\ 2.52 & -1.63 & -1.63 \\ -2.52 & -1.63 & -1.64 \end{pmatrix} \quad \textbf{(11)}$$

$$I_1^{-1} = \begin{pmatrix} 0.0002 & 0.00023 & 0.00005 \\ 0.00023 & 0.00013 & -0.00011 \\ 0.00005 & -0.00011 & -0.000092 \end{pmatrix} \quad \textbf{(12)}$$

and

$$I_3^{-1} \begin{pmatrix} -1.37 \text{ x } 10^{-13} & -4.91 \text{ x } 10^{15} \\ -4.91 \text{ x } 10^{-15} & -1.77 \text{ x } 10^{-5} \end{pmatrix} \quad \textbf{(13)}$$

where, $\hat{\sigma}_1$=-1.63, $\hat{\sigma}_2$=0.00013, and $\hat{\sigma}_3 = -1.77 \text{ x } 10^{15}$ are the respective scale parameter estimate values for each of the distributions. The least scale parameter value being $\hat{\sigma}_1 = -1.63$.

CONCLUSION AND RECOMMENDATION

It is obvious that the Frechet distribution gave the least scale parameter estimate (standard deviation). It then follows that our data can best be modeled by the Frechet distribution.

We therefore recommend that for accurate predictions or forecasts to be attained, minimum value measurements of temperature, pressure, humidity, and other elements of weather and climate should be modeled by the Frechet distribution.

This paper therefore provides meteorologists with a strong basis of modeling and analysis of minimum-value observations and consequently error-free weather forecasts.

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