Solution of Non-Linear Volterra Integral Equations by Chebyshev Collocation Approximation Method

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ABSTRACT

This paper presents a solution technique for the Non-linear Volterra Integral Equations (NLVIEs) using Chebyshev Collocation Approximation Method (CCAM). The first procedure of this technique is the transformation of the NLVIEs into the linear Volterra integral equations (LVIEs) which are integrated via Chebyshev collocation points to produce algebraic linear equations which are solved with Maple 18 software to obtain the unknown constants. Some examples are presented to illustrate the efficiency and simplicity of the method, the results obtained are compared with some of the existing methods in the literature.

(Keywords: collocation method, nonlinear integral equations, approximation method, linear Volterra integral equations)

(2010 Mathematics Subject Classification: 65L60; 41A50)

INTRODUCTION

Many alternative methods have been used to approximate the solution of a system of integral equations in recent years. Because few of these equations can be solved explicitly, numerical approaches involving appropriate combinations of numerical integration and interpolation are frequently used (Baker, 2000; Oyedepo, et al., 2019; Ishola, et al., 2022; Adebisi, et al., 2021; Uwaheren, et al., 2021; and Peter, 2020). Other numerical methods can be found in (Peter, et al., 2022; Peter, et al., 2021; Ayoade, et al, 2020; and Peter, et al., 2018).

There are two distinct but related ways to formulate many problems in mathematics, particularly in applied mathematics; these are, differential equations or as integral equations. Though, it is a known fact that there is a close relationship between differential equation and integral equation, much emphasis has been placed in the solution of differential equations over the years more than the solutions of integral equations for an agreed reason that the latter are a bit more difficult and tedious to solve than the former (Taiwo and Ishola, 2009).

Volterra integral equations appear in a variety of scientific applications, and they can also be derived from an initial value problem (Wazwaz, 2011 and Hamoud, et al., 2019). Linear and nonlinear Volterra integral equations are well-known in many scientific disciplines, including population dynamics, epidemic spread, and semi-conductor devices (Hamoud and Ghadle, 2018).

Over the past fifty years, substantial progress in developing analytical and numerical solutions of integral equations of different kinds; linear and nonlinear cases have been considered (Basseem, 2015). Non-linear phenomena are of fundamental importance in various fields of science and engineering. The non-linear models of real-life problems are still difficult to solve either numerically or theoretically. There has recently been a lot of focus on finding better and more efficient methods for determining a solution, whether approximate or exact, analytical or numerical, to nonlinear models. (Batiha, *et al.*, 2008).

Rahman, 2007 stated that non-linear integral equations yield a considerable amount of difficulties, however, due to recent development of novel techniques it is now possible to find solutions of some types of non-linear integral equations if not all. In general, the solution of the non-linear integral equation is not unique. However, the existence of a unique solution of non-linear integral equations with specific conditions is possible, for instance, Mirzaei (2010) revealed the effectiveness of homotopy perturbation method in solving the second kind of non-linear integral equations. Babolian, *et al.* (2007) presented a numerical method for solving Fredholm-Volterra Hammerstein integral equations in which the unknown function replaced by truncated series of well-known Chebyshev expansion of functions.

Yang (2021) adopted the Chebyshev collocation method to find an approximate solution for nonlinear integral equations. Taiwo and Abubakar (2013) obtained the numerical solutions of linear integro-differential equations using integral collocation approximation methods. Also, (Abubakar and Taiwo, 2014) used integral collocation approximation methods to solve high-orders linear Fredholm-Volterra integro-differential equations. Agbolade and Anake (2017) numerically solved linear Volterra integro-differential equations by collocation methods. Cardone, *et al.* (2018) stated that collocation methods have several desirable properties and that they provide an approximation over the entire integration interval to the solution of the equation, which reveals to be quite useful in a variable-stepsize implementation.

Other good properties of collocation methods are their high order of convergence, strong stability properties and flexibility. Ming and Huang (2017) derived and analysed a solution representation for Volterra functional integral equations using collocation at the m-Gauss-Legendre points and the m-Radau II points. In this work, two collocation points namely: Chebyshev Gauss-Lobatto Collocation Points (CGLCP) and Chebyshev Gauss-Radau Collocation Points (CGRCP) would be employed for the numerical solution of NLVIE of the form:

$$u(x) = g(x) + \lambda \int_a^x K(x, s, u(s))ds \quad \textbf{(1)}$$

where $u(x)$ is an unknown function, a is a real constant, and the functions $g(x)$ and $K(x, s, u(s))$ are analytical on R and R^3, respectively. λ is a scalar parameter.

Linearization of the NLVIE

For the nonlinear function in 1 above, $K(x, s, u(s))$ can be approximated by the first three terms of its Taylor series expansion in two ways:

a) If K is regular, around (x_n, s_n, u_n) to obtain:

$$K(x, s, u) = K(x_n, s_n, u_n) + (x - x_n)\frac{\delta}{\delta x}K(x_n, s_n, u_n) + (s - s_n)\frac{\delta}{\delta s}K(x_n, s_n, u_n) + (u - u_n)\frac{\delta}{\delta u}K(x_n, s_n, u_n) \quad \textbf{(2)}$$

substitute (2) into (1) to obtain:

$$u(x) = g(x) + \lambda \int_a^x [K_n + (x - x_n)\alpha_n + (s - s_n)\beta_n + (u - u_n)\gamma_n]ds \quad \textbf{(3)}$$

where,

$$\begin{aligned} u(x_n) &= u_n \\ K_n &= K(x_n, s_n, u_n) \\ \alpha_n &= \frac{\delta}{\delta x}K(x_n, s_n, u_n) \\ \beta_n &= \frac{\delta}{\delta s}K(x_n, s_n, u_n) \\ \gamma_n &= \frac{\delta}{\delta u}K(x_n, s_n, u_n) \end{aligned} \quad \textbf{(4)}$$

In the integral part of 3, s is an independent variable, u is a dependent variable and x is a parameter, therefore by integrating it with respect to s, we have:

$$u(x) = g(x) + \lambda\gamma_n \int_a^x u(s)ds + \lambda[K_n + (x - x_n)\alpha_n - u_n\gamma_n] \int_a^x ds + \lambda\beta_n \int_a^x (s - s_n)ds \quad \textbf{(5)}$$

Further simplification gives:

$$u(x) = g(x) + \lambda[K_n + (x - x_n)\alpha_n - u_n\gamma_n](x - a) + \frac{\lambda}{2}[(x - s_n)^2 - (a - s_n)^2]\beta_n + \lambda\gamma_n \int_a^x u(s)ds \quad \textbf{(6)}$$

Equation 6 is the linearized form of 1.

b) If $K(x,s,u(s))$ is singular at x_n, the above derivation is not valid, but $K(x,s,u)$ in 1 may also be approximated around (x_{n+1},s_{n+1},u_n) by:

$$u(x) = g(x) + \lambda[K_{n+1} + (x - x_{n+1})\alpha_{n+1} - u_n\gamma_{n+1}](x-a)$$

$$+\frac{\lambda}{2}[(x-s_{n+1})^2 - (a-s_n)^2]\beta_{n+1} + \lambda\gamma_{n+1}\int_a^x u(s)ds \qquad \textbf{(7)}$$

where

$$\begin{aligned} u(x_n) &= u_n \\ K_{n+1} &= K(x_{n+1},s_{n+1},u_n) \\ \alpha_{n+1} &= \frac{\delta}{\delta x}K(x_{n+1},s_{n+1},u_n) \\ \beta_{n+1} &= \frac{\delta}{\delta s}K(x_{n+1},s_{n+1},u_n) \\ \gamma_{n+1} &= \frac{\delta}{\delta u}K(x_{n+1},s_{n+1},u_n) \end{aligned} \qquad \textbf{(8)}$$

Method of solution by Chebyshev Collocation Points

This section assumes an approximation of the form:

$$u(x) \approx u_{n+1,N}(x) = \sum_{r=0}^{N} \mu_{n,r}T_r(x) \qquad \textbf{(9)}$$

where n is the number of iterations; $\mu_{n,r}$ are to be determined and $T_r(x)$ are the Chebyshev polynomials of degree r. Chebyshev polynomials are well-known family of orthogonal polynomials on the interval $[-1,1]$ and have many applications. They are widely used because of their good properties in the approximation of functions (Ramosa and Vigo-Aguiar, 2007).

Chebyshev polynomials are employed as basis functions to approximate the solutions of several numerical problems involving integral and integro-differential equations. Chebyshev polynomials $T_r(x)$ of the first kind valid in $[-1,1]$ is defined as:

$$T_r(x) = cos(rcos^{-1}x) \qquad \textbf{(10)}$$

and satisfies the recurrence relation:

$$T_{r+1}(x) = 2xT_r(x) - T_{r-1}(x) \quad ; \quad n \geq 1 \qquad \textbf{(11)}$$

The shifted Chebyshev polynomial valid in $[a,b]$ is defined as:

$$T_r(x) = cos\left[r\ cos^{-1}\left(\frac{2x-b-a}{b-a}\right)\right]\ ;\ r \geq 0 \qquad \textbf{(12)}$$

and satisfies the recurrence relation:

$$T_{r+1}(x) = 2\left(\frac{2x-b-a}{b-a}\right)T_r(x) - T_{r-1}(x); \qquad a \leq x \leq b \qquad \textbf{(13)}$$

Linearization of nonlinear equations always lead to the use of iterations, therefore we substitute Equation 9 into the linearized Equation 6 to obtain:

$$\sum_{r=0}^{N}\mu_{n,r}T_r(x) = g(x) + \lambda[K_n + (x - x_n)\alpha_n - u_n\gamma_n](x-a)$$

$$+\frac{\lambda}{2}[(x-s_n)^2 - (a-s_n)^2]\beta_n + \lambda\gamma_n\int_a^x \sum_{r=0}^{N}\mu_{n,r}T_r(s)ds \qquad \textbf{(14)}$$

The integral part of equation 14 is evaluated such that the variable s changes to x and the resulting equation is the collocated using the following two collocation points:

i) Chebyshev Gauss-Lobbato Collocation Points defined by:

$$x_i = cos(\frac{\pi}{N})\ ;\ i = 0,1,2,\ldots N \qquad \textbf{(15)}$$

These points generate $(N+1)$ algebraic linear equations with $(N+1)$ unknown constants, $\mu_{n,r}$ $(r = 0,1,2,\ldots N)$, which are solved by Maple 18.

If $n = 0$, the solution is obtained at the first iteration and $u_{0,N}$ serves as the initial guess.

ii) Chebyshev Gauss-Radau Collocation Points defined by:

$$x_i = cos(\frac{\pi j}{N+1}) \ ; \ j = 0,1,2,...N \quad \textbf{(16)}$$

These points produce $(N+1)$ algebraic linear equations with $(N+1)$ unknown constants, $\mu_{n,r}$ $(r = 0,1,2,...N)$, which are solved by Maple 18.

Numerical Examples

This section presents the application of Chebyshev collocation approximation method on some examples after transforming the NLVIEs into LVIEs. All algebraic computations are executed using Maple18 software package.

Example 1:
Consider the non-linear Volterra integral equation:

$$u(x) = sec(x) + tan(x) + x - \int_0^x (1 + u(s)^2)ds \quad \textbf{(17)}$$

with exact solution $u(x) = sec(x)$.

Following the linearization techniques discussed earlier, equation will be transformed to the LVIE of the form:

$$u(x) = sec(x) + tan(x) - xu_n^2 + 2u_n \int_0^x (u(s))ds \quad \textbf{(18)}$$

Table 1 shows the numerical results for $n = 4$ at the sixth iteration and comparison with the exact solution.

Example 2:
Consider the non-linear Volterra integral equation"

$$u(x) = e^x + \frac{1}{2}x(e^{2x} - 1) - \int_0^x (xu(s)^2)ds \quad \textbf{(19)}$$

with exact solution $u(x) = e^x$.

By linearization method, Equation (19) is reduced to its linear form thus:

$$u(x) = e^x + \frac{1}{2}x(e^{2x} - 1) - 2x^2u_n^2 - 2xu_n \int_0^x u(s)ds \quad \textbf{(20)}$$

Table 2 shows the numerical results for $n = 4$ at the sixth iteration and comparison with the exact solution.

CONCLUSION

By introducing the linearization technique, non-linear integral equations in applied sciences and physics can be transformed into their forms which may then be integrated using classical methods. In this work, both the applicability and the effectiveness of collocation methods for the solution of NLVIEs have been examined by means of some numerical examples. We plan to demonstrate that the proposed methods may be applied to coupled Volterra integro differential equations in the future.

Table 1: Comparison of Numerical Solution and Exact Solution for Example 1.

x	Exact	CGLCP	CGRCP
(0.0)	0.000000000	0.0000000012	0.0000000014
(0.1)	1.005020918	1.0049867767	1.0050113273
(0.2)	1.020338845	1.0199026841	1.0201126039
(0.3)	1.046751602	1.0459846657	1.0466521391
(0.4)	1.085704428	1.0858528053	1.0858216834
(0.5)	1.139493927	1.1398214367	1.1398221982
(0.6)	1.211628315	1.2105974411	1.2106892278
(0.7)	1.307459260	1.3087310427	1.3089646542
(0.8)	1.435324200	1.4353836226	1.4352894694
(0.9)	1.608725810	1.6082020385	1.6083019387
(1.0)	1.850815718	1.8504456984	1.8505321170

Table 2: Comparison of Numerical Solution and Exact Solution for Example 2.

x	Exact	CGLCP	CGRCP
(0.0)	1.000000000	1.0002582663	1.0002485386
(0.1)	1.105170918	1.1049757984	1.1051735486
(0.2)	1.221402758	1.2219026418	1.2216987349
(0.3)	1.349858808	1.3496869572	1.3496927537
(0.4)	1.491824698	1.4909553863	1.4917688411
(0.5)	1.648721271	1.6398214367	1.6397802833
(0.6)	1.822118800	2.8214617168	2.8230017532
(0.7)	2.013752707	2.0134728762	2.0134643994
(0.8)	2.225540928	2.2247932713	2.2245645518
(0.9)	2.459603111	2.4682291682	2.4682393763
(1.0)	2.718281828	2.7089674849	2.7135228231

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SUGGESTED CITATION

Ishola, C.Y., O.A. Taiwo, K.A. Adedokun, and M.O. Ajisope. 2022. "Solution of Non-Linear Volterra Integral Equations by Chebyshev Collocation Approximation Method". *Pacific Journal of Science and Technology*. 23(1): 34-39.

Pacific Journal of Science and Technology