Bernstein Modified Homotopy Perturbation Method for the Solution of Volterra Fractional Integro-Differential Equations

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ABSTRACT

This study gears towards finding a numerical solution to fractional integro-differential equations. The techniques involve the application of the homotopy perturbation method and using Bernstein polynomials as the initial approximation. The resulting equations involve comparing the coefficients of the homotopy parameter P, which then resulted in a system of linear algebraic equations by which the solution of the unknown constants were obtained utilizing Maple 18. A few examples are introduced to show the exactness of the techniques. The outcomes are thought about, it may be said that the proposed technique performed precisely better than other techniques since the exact solution is found for N=2.

(Keywords: modified homotopy perturbation method, Bernstein polynomials, fractional integro-differential equations)

INTRODUCTION

Fractional calculus is a field dealing with integral and derivatives of arbitrary orders, and their applications in science, engineering and other fields. The idea is from the ordinary calculus. According to Adam (2004) and Caputo (1967) it was discovered by Leibniz in the year 1695 a few years after he discovered ordinary calculus but was later forgotten due to the complexity of the formula.

Many real-world physical problems can be models by fractional integro-differential equations (FIDEs) including the modeling of the earthquakes, diminishing the spread of viruses, control of the memory behavior of electric sockets, and many other examples. There are many intriguing books about fractional calculus and fractional differential equations (Caputo,1967; Munkhammar, 2005; and Kumar, et al., 2017).

Since the fractional calculus has pulled in substantially more enthusiasm among mathematicians and different researchers, the solutions of the FIDEs have been concentrated habitually as of late. Many FIDEs cannot be solved analytically, and hence finding good approximate solutions, using numerical techniques, will be very helpful. Several numerical methods that are used to find the solutions of the FIDEs are given below.

Awawdeh, et al. (2011) introduced homotopy analysis method for the solution of FIDEs. Mohammed (2014) employed LSM for solving FIDEs using shifted Chebyshev polynomial of the first kind as the basis function. Sahu and Saha (2016) also introduced a method called "A novel Legendre wavelet Petrov–Galerkin Method (LWPGM) for FIDEs". Dilkel and Aysegül (2018) applied collocation method using Laguerre Polynomials as the basis functions. Oyedepo et al. (2019) employed Homotopy perturbation and LSM for solving FIDEs. Several numerical methods to solve the FIDEs have be found in Ibrahim et al. (2017), Ayoade, et al. (2018), Peter et al. (2018), Peter (2020), and Peter et. al. (2021).

However, there has not been a method in the literature for FIDEs in terms of modified homotopy perturbation method using Bernstein polynomials as the initial approximation. That is why, in this study, a method called modified homotopy perturbation method is presented to find the solutions of FIDEs in the form:

$$D^{\alpha}u(x) = p(x)u(x) + f(x) + \int_0^x k(x,t)u(x)dt, \quad o \leq x,t \leq 1, \tag{1}$$

With the following supplementary conditions:

$$u^{(j)}(0) = \delta_j, j = 0,1,2,\dots,m-1, m-1 < \alpha \leq m, \ m \in, N$$

Where $D^{\alpha}u(x)$ indicates the $\propto th$ Caputo fractional derivative of $u(x)$; $p(x)$,$f(x)$, $K(x,t)$ are given smooth functions, δ_j are real constant, x and t are real variables varying [0, 1] and $u(x)$ is the unknown function to be determined.

SOME RELEVANT BASIC DEFINITIONS

Definition 1:
Fraction Calculus involves differentiation and integration of arbitrary order (all real numbers and complex values). Example $D^{\frac{1}{2}}, D^{\pi}, D^{2+i}$ e.t.c

Definition 2:
The Caputor Factional Derivative is defined as:

$$D^{\alpha}f(x) = \frac{1}{\Gamma(n-\propto)}\int_0^x (x-s)^{n-\propto-1} f^{m}(s)ds \tag{2}$$

Where m is a positive integer with the property that $n-1 < \propto < n$.

For example if $0 < \propto < 1$ the caputo fractional derivative is:

$$D^{\alpha}f(x) = \frac{1}{\Gamma(1-\propto)}\int_0^x (x-s)^{-\propto} f^{1}(s)ds$$

Hence, we have the following properties:

(1) $J^{\alpha} J^{v} f = j^{\alpha+v} f, \alpha, v > 0, f \in C_{\mu}, \mu > 0$

(2) $J^{\alpha}x^{\gamma} = \frac{\Gamma(\lambda+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}, \alpha > 0, \gamma > -1, x > 0$

(3) $J^{\alpha} D^{\alpha} f(x) = f(x) - \sum_{k=0}^{n-1} f^{k}(0)\frac{x^k}{k!}, \qquad x > 0, n-1 < \alpha \leq n$

(4) $D^{\alpha} J^{\alpha} f(x) = f(x), \quad x > 0, n-1 < \alpha \leq n,$

(5) $D^{\alpha}C = 0, C$ is the constant,

(6) $\begin{cases} 0, & \beta \in N_0, \beta < [\alpha], \\ D^\alpha x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha}, & \beta \in N_0, \beta \geq [\alpha], \end{cases}$

Where $[\alpha]$ denoted the smallest integer greater than or equal to α and $N_0 = \{0,1.2, \ldots\}$

Definition 3:
Bernstein basis polynomials: A Bernstein polynomial of degree N is defined by:

$$B_{i,m}(x) = \binom{m}{i} x^i (1-x)^{m-i} \quad i = 0, 1 \ldots n, \tag{3}$$

where,
$\binom{m}{i} = \frac{m!}{i!(m-1)!}$

Often, for mathematical convenience, we set $B_{i,m}(x) = 0$ if < 0 or $j > m$

Definition 4:
Bernstein polynomials: A linear combination Bernstein basis polynomials:

$$u_m(x) = \sum_j^m a_j u_j(x) \tag{4}$$

The Bernstein polynomial of degree n where $a_{j,}$ $j = 0,1,2, \ldots \ldots$ are constants.

Examples:
The first few Bernstein basis polynomials are:

$u_0(x) = 1, u_1(x) = a_0(1-x) + a_1 x,\ u_2(x) = a_0(1 - 2x + x^2) + a_1(2x - 2x^2) + a_2 x^2$

Definition 5:
In this work, we defined absolute error as:

Absolute Error $= |U(x) - u_m(x)|; \quad 0 \leq x \leq 1,$ **(5)**

where $U(x)$ is the exact solution and $u_m(x)$ is the approximate solution.

where $u_m(x)$ Bernstein polynomial of degree m where a_j, $\mathrm{j} = 0,1,2, \ldots$ are constants.

MODIFIED HOMOTOPY PERTURBATION METHOD (MHPM)

Consider (1) operating with J^α on both sides as follows:

$$u(x) = \sum_k^{m-1} u^k(0) \frac{x^k}{k!} + J^\alpha \left[p(x)u(x) + f(x) + \int_0^x k(x,t)u(t)dt\right] \tag{6}$$

We solved (6) by the homotopy perturbation method, we constructed the following homotopy:

$$H(U(x),p) = (1-p)(U(x)-u_0(x)) + p[U(x) - \sum_k^{m-1} u^k(0)\frac{x^k}{k!} - J^\alpha\,[p(x)u(x) - f(x) - \int_0^x k(x,t)u(t)dt \quad (7)$$

Where $p \in [0,1]$ is called the homotopy parameter, and $u_0(x)$ is an initial approximation to the solution of (7). Simplifying (7), to get:

$$U(x) = u_0(x) \; - pu_0(x) \; + p\left[\sum_k^{m-1} u^k(0)\frac{x^k}{k!}\right] + J^\alpha[p(x)U(x)] + p[J^\alpha f(x)] + p[J^\alpha(\int_0^x k(x,t)u(t)dt)] \quad (8)$$

Suppose the solution of (8) have the form:

$$U(x) = \sum_{i=0}^n p^i U_i(x) \quad (9)$$

Where $U_i(x)$,i $= 0,1,2,3,...$ are functions to be determined?

Now suppose that the initial approximation to the solution $u_0(x)$ has the form:

$$u_0(x)=\sum_{i=0}^n a_i u_i^*(x) \quad (10)$$

Where a_i are unknown coefficient of $u_0(x), u_1(x), u_2(x), ...$ are specific functions depending on the problem.

Substituting (10) into (9) to have:

$$\sum_{i=0}^n p^i U_i(x) = u_0(x) \; - pu_0(x) \; + p\left[\sum_k^{m-1} u^k(0)\frac{x^k}{k!}\right] + J^\alpha\left[p(x)\sum_{i=0}^n p^i U_i(x)\right] + p[J^\alpha f(x)] + p[J^\alpha(\int_0^x k(x,t)\sum_{i=0}^n p^i U_i(t)\,dt)] \quad (11)$$

Equating the like power of p we obtain the following:

$$p^0: U_0(x) = u_0(x) \quad (12)$$

$$p^1: U_1(x) = -u_0(x) + \sum_k^{m-1} u^k(o^+)\frac{x^k}{k!} + J^\alpha[p(x)U_0(x)] + J^\alpha f(x) + J^\alpha(\int_0^x k(x,t)U_0(t)dt) \quad (13)$$

$$p^2: U_2(x) = J^\alpha[p(x)U_1(x)] + J^\alpha(\int_0^x k(x,t)U_1(t)dt) \quad (14)$$

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$$p^{n+1}: U_{n+1}(x) = J^\alpha[p(x)U_n(x)] + J^\alpha(\int_0^x k(x,t)U_n(t)dt) \quad (15)$$

Now if these equations are solved in a way that $U_1(x) = 0$, then $(14 - 15)$ resulted to $U_2(x) = U_3(x) = \cdots = 0$. Therefore the solution can be obtained by substituting (10) into (13) to get:

$$U_0(x)=\sum_{i=0}^n a_i u_i^*(x) \quad (16)$$

Also, substituting (10) and (16) into (15) to have:

$$-\sum_{i=0}^{n} a_i u_i^*(x) + \sum_{k}^{m-1} u^k(0)\frac{x^k}{k!} + J^\alpha[p(x)\sum_{i=0}^{n} a_i u_i^*(x)] + J^\alpha f(x) + J^\alpha\left(\int_0^x k(x,t)\sum_{i=0}^{n} a_i u_i^*(t)\,dt\right) = 0 \quad \textbf{(17)}$$

Simplifying (17) further gives algebraic linear systems of equations by equating corresponding coefficients of various powers of the independent variable in (17) . The systems of equations are then solved using maple 18. The unknown constants obtained are then substituted back into (10) to give the required approximate solution.

NUMERICAL EXAMPLES

Example 1: Consider the following fractional Integro-differential (Awawdeh, et.al. 2011)

$$D^{0.75}u(x) = \frac{1}{\Gamma(1.25)}x^{0.25} + (x\cos x - \sin x)u(x) + \int_0^x x\sin t\, u(t)dt \quad \textbf{(18)}$$

Subject to $u(0) = 0$ with the exact solution $u(x) = x$.

Applying the recommended technique above with the guide of Bernstein polynomials as the initial approximation to FIDEs (18). The following constants were obtained a=0, b= 0.5 c= 1, which were then substituted into the initial approximation and thereafter we obtained the exact solution. Homotopy analysis technique was applied to find the numerical solution of a similar problem by Awawdeh, et al. (2011). The approximate solution was found at N= 5, however, the numerical results of the errors of the technique were not stated. Also Sahu and Saha (2016) found the approximate solution with the maximum absolute error 1.1×10^{-16} .

Example 2: Consider the following fractional Integro-differential (Awawdeh, et al. 2011).

$$D^{\frac{1}{2}}u(x) = u(x) + \frac{8x^{2.25}}{3\Gamma(0.5)} - x^2 - \frac{1}{3}x^3 + \int_0^x tu(t)dt \quad \textbf{(19)}$$

Subject to $u(0) = 0$ with the exact solution $U(x) = x^2$.

Also as in example 1, The solution obtained solving (19) using the suggested technique is in great concurrence with the already exact solution. Homotopy analysis technique was applied to find the numerical solution of a similar problem by Awawdeh, et al. (2011). The approximate solution was found at N= 5, however, the numerical results of the errors of the method were not stated. Also, [10] found the approximate solution with the maximum absolute error 4.2×10^{-15} by the LWPGM for N= 6. Looking at the outcomes of the results, it tends to be said that our method performed more precisely than different techniques since the exact solution is found for N=2.

Example 3: Consider the following fractional Integro-differential (Awawdeh, et al. 2011).

$$D^{\sqrt{3}}u(x) = \frac{1}{\Gamma(3-\sqrt{3})}x^{2-\sqrt{3}} + 2\sin x - 2x + \int_0^x \cos(x-t)\, u(t)dt \quad \textbf{(20)}$$

Subject to $u(0) = 0$, $u'(0) = 0$ with exact solution $U(x) = x^2$.

So also as in example 1 and 2, applying the solution method above with the aid of Bernstein polynomials as the initial approximation to FIDEs (20). The following constants were obtained a=0, b= 0 c= 1, which were then substituted into the initial approximation and thereafter we obtained the exact solution. This problem was also solved by Awawdeh, et al. (2011) and they found the approximate solution by the homotopy analysis method for N=5. Looking at the outcomes of the results, evidently, it tends to be said that our method performed more precisely than the other techniques since the exact solution is obtained at N=2.

CONCLUSION

This study has focused on developing a simple procedure to obtain an approximate solution of FIDEs. The suggested method, modified homotopy perturbation method using Bernstein polynomials as the initial approximation is presented for the first time in the literature. The results obtained compared with Awawdeh, et. al. (2011) and Sahu and Saha (2016) showed that the suggested method is more accurate than other techniques in the literature, and better than Awawdeh, et. al. (2011) and Sahu and Saha (2016). Hence, calculation showed that the suggested method is an efficient technique in finding an excellent solution for this type of equation (1).

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