Extension of Transmuted HalfNormal Distribution Properties and Its Application

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ABSTRACT

This research extended the HalfNormal distribution to Transmuted HalfNormal Distribution (THND) and it generalizes the classical HalfNormal by expanding its scope to modeling high-class random process that cannot be easily modeled with existing probability distribution. The THND was compounded and its statistical properties were obtained such as moment, hazard function, reliability, probability density and cumulative distribution function. This hybrid model can be used to capture non-normal data with highly skewed, heavily tailed and Leptokurtic distribution.

(Keywords: transmuted halfnormal distribution, moment, reliability function, hazard function, cumulative distribution function).

INTRODUCTION

In many applied sciences such as medicine, engineering, and finance, modeling and analyzing lifetime data is imperative. Many lifetime distributions have been used to model such type of phenomena from which the data was generated. In statistics, it is crucial to understand the underlying probability distribution or phenomena which the available data followed before deciding the appropriate statistical test to be employed in analyzing the data.

In this era of advancement in science and technology, many processes have become complex to the extent that it becomes difficult to accurately model the stochastic behavior of such processes using those classical probability distribution. Since the quality of the procedures used in a statistical analysis depends on the assumed probability model, then there is a need to construct new probability distribution that can capture the pattern of such processes and use such distribution to construct necessary statistical tests, confidence intervals, and predictions regarding the subsequent behavior of such phenomena. However, there remain many important problems where the real data does not follow any of the classical or standard probability models (existing distribution), hence, the need for mixing families of distributions.

HalfNormal distribution was used to model Brownian movement and can also be used in the modeling of measurement data and lifetime data. Let $X \sim N(0,\sigma^2)$, then $Y = |X|$ follows HalfNormal distribution. The HalfNormal is a fold at the mean of an ordinary normal distribution with mean zero, where σ is the scale parameter. By obtaining the transmuted version of HalfNormal distribution, the resulting hybrid distribution is called transmuted HalfNormal distribution.

This proposed mixed distribution has more parameters as compared to its respective parent distributions (HalfNormal) and it has wider applicability exceeding modeling particular size but in modeling many stochastic processes and stochastic phenomena which cannot be easily modeled by one parameter probability density (parent distribution) such as disease growth, epidemiological studies of disease, buying behavior of consumers towards certain economic product, etc. It is in this view that this research is structured to propose new hybrid distributions (Transmuted HalfNormal) with a view to studying its properties and application to real life data to reflect the flexibility, stability and consistency of this hybrid model as compared to its parent distributions.

LITERATURE REVIEW

Many researchers have worked on aspects of compounding two or more probability distributions to obtain a family of hybrid distributions which are more efficient than their parent distributions due to the addition of more parameters which increases the flexibility of the mixture of distributions in tracking many random phenomena that cannot be easily modeled by their parent distributions.

Many authors have also worked on compounding beta distributions with other distributions. The beta family of distributions became popular some years back, and include Beta-Normal (Eugene and Famoye, 2002); Beta-Gumbel (Nadarajah and Kotz, 2004), Beta-Weibull (Famoye, Lee, and Olugbenga, 2005), Beta-Exponential (Nadarajah and Kotz, 2006); Beta-Rayleigh (Akinsete and Lowe, 2009); Beta Fre´chet (Nadarajah and Gupta, 2004), Beta-HalfNormal (Akomolafe and Maradesa, 2017), Beta-Gamma; Beta-F, Beta-T; Beta-Beta; Beta-Modified Weibull; Beta-Nakagami; and others.

Cordeiro and de Castro discussed moment generating function of for generated beta distribution. When we consider σ as a random variable, the bayes convert our belief about the parameter σ of HalfNormal distribution (before seeing data) into posterior probability, P($\sigma|X|$) , by using the likelihood function P(X| σ) .The maximum a-posteriori (MAP) estimate is defined as:

$$\sigma = \ argmax\ P(\sigma|X) = argmax_{\sigma}\ \frac{P(\sigma).P(X|\sigma)}{P(X)}$$

Having used the Maximum A Posteriori Estimation and Maximum Likelihood Estimation, the mathematical approach suggested that the Maximum A Posteriori Estimation is a better fit as compared to the maximum likelihood estimation.

Lifetime data can be modeled using several existing distributions. However, some of these lifetime data do not follow these existing distributions or are inappropriately described by them. Hence, the need to develop distributions that could better describes some of these phenomena and provide greater flexibility in the modeling of lifetime data than the baseline distributions.

Transmuted family of distributions has many useful applications in the field of medicine, ecology, and reliability analysis. The need to apply this approach to HalfNormal distribution arose because of the curiosity related to extending the scope of application and to provide a more robust modeling approach to capture the behavior of stochastic phenomena. In this sense, it would be useful to capture data that has a non-normal distribution, is heavily tailed, and is leptokurtic in nature.

Statement of the Problem

With advancing time, many processes tend to be more complex. As an example, different diseases keep surfacing and becoming new points of research in epidemiology. With advancing complexities of life, the existing probability model will be unfit to model many complex processes in the future. There is a need for mathematical modelers to compound the multi-parameters distribution (hybrid) with more flexibility in tracking the statistical behavior of stochastic processes. It is on this note that it becomes imperative to obtain the transmuted version of HalfNormal distribution so as to add transmuted parameters as well as providing a comprehensive description of the distribution for modeling reliability data. The additional parameter makes it easy for the hybrid distribution to have wider applicability (flexibility) than the classical distribution (HalfNormal).

Source of Data

The data used for this study is primary data collected from the standard wholesale outlet of GLO Communication Outlet located in Lagos and Ibadan, Nigeria where retailers come to buy data in bulk depicting the data consumption of people that use GLO network for internet browsing. Their purchasing pattern is stochastic in nature after subjecting the data collected for twelve consecutive weeks to some conventional tests. The data is called GLO-DATA in the study.

METHODOLOGY

A random variable X is said to have a transmuted distribution if its distribution function is given by:

$$f(x) = (1 + \lambda)g(x) - 2\lambda g(x) \qquad |\lambda| \leq 1$$
$$= g(x) + \lambda g(x) - 2\lambda g(x) \quad \textbf{(1)}$$

And the density function is given by:

$$F(x) = (1 + \lambda)G(x) - \lambda(G(x))^2 \quad \textbf{(2)}$$

$$F(x) = G(x)[(1 + \lambda) - \lambda G(x)] \quad \textbf{(3)}$$

Where $g(x) = \frac{\sqrt{2}}{\sigma\sqrt{\pi}} e^{-\frac{x^2}{2\sigma^2}}$ and $G(x) = \text{erf}(\frac{x}{\sigma\sqrt{2}})$ are the cumulative distribution and density function of the parent distribution (HalfNormal). The parent distribution can be any probability distribution from which we want to obtain its transmuted version.

From (1) and (3), the pdf and cdf of the Transmuted HalfNormal can be obtained shown by (4) and (5) respectively:

$$f_{THND}(\text{x}, \sigma, \lambda) = \left[(1 + \lambda) - 2\lambda\text{erf}\left(\frac{x}{\sigma\sqrt{2}}\right)\right] \frac{\sqrt{2}}{\sigma\sqrt{\pi}} e^{-\frac{x^2}{2\sigma^2}} \quad \textbf{(4)}$$

We can expand (4) to yield (5):

$$= \frac{\sqrt{2}}{\sigma\sqrt{\pi}} e^{-\frac{x^2}{2\sigma^2}} + \lambda \frac{\sqrt{2}}{\sigma\sqrt{\pi}} e^{-\frac{x^2}{2\sigma^2}} - 2\lambda\, \text{erf}\left(\frac{x}{\sigma\sqrt{2}}\right) \quad \textbf{(5)}$$

$$F_{THND}(\text{x}, \sigma, \lambda) = (1 + \lambda)\left(\text{erf}\left(\frac{x}{\sigma\sqrt{2}}\right)\right) - \lambda(\text{erf}(\frac{x}{\sigma\sqrt{2}}))^2 \quad \textbf{(6)}$$

Estimation of Parameter (Maximum Likelihood Method)

Using the maximum likelihood estimation technique, we estimate the parameter of the hybrid distribution. Given that $f_{THND}(x; \lambda, \sigma)$ is the pdf of THND, then the likelihood function is given by:

$$L(f_{THND}(x; x; \lambda, \sigma) = \prod_{i=1}^{n} \left[(1 + \lambda) - 2\lambda\text{erf}\left(\frac{x}{\sigma\sqrt{2}}\right)\right] \frac{\sqrt{2}}{\sigma\sqrt{\pi}} e^{-\frac{x^2}{2\sigma^2}} \quad \textbf{(7)}$$

By taking the natural logarithm of (5) and obtaining the derivatives with respect to each of the parameters, we can obtain the estimate of those parameters when setting the derivative to zero and solve the equations.

Investigation of Some Properties of the Distribution

Certain descriptive properties of the proposed distribution will be verified using mathematical and graphical approach and other methods such as classical method of moment generating function and others. Among the properties to be investigated are:

Moment

$$\boldsymbol{Ex^r} = \int_{\mathbf{0}}^{\infty} \boldsymbol{x^r} f_{THND}(x; \lambda, \sigma)\, \mathrm{dx} \quad \textbf{(8)}$$

$$= (1+\lambda)\frac{\sqrt{2}}{\sigma\sqrt{\pi}}\int_{\mathbf{0}}^{\infty} \boldsymbol{x^r} e^{-\frac{x^2}{2\sigma^2}} - \mathbf{2}\lambda\frac{\sqrt{2}}{\sigma\sqrt{\pi}}\int_{0}^{\infty} \boldsymbol{x^r}\ \mathrm{erf}\left(\frac{x}{\sigma\sqrt{2}}\right) e^{-\frac{x^2}{2\sigma^2}} dx \quad \textbf{(9)}$$

erf (x) can be expressed in term of confluence hypergeometric function of the first kind:

$$= \frac{2x}{\sqrt{\pi}}\, M\left(\frac{1}{2}, \frac{3}{2}, -x^2\right) \qquad = \frac{2x}{\sqrt{\pi}}\, e^{-x^2} M\left(1, \frac{3}{2}, x^2\right)$$

Therefore

$$\frac{d^n}{dx^n} = (-1)^n \frac{2}{\sqrt{\pi}}\ H_{n-1}(x)\, e^{-x^2} \quad ; where\ H_{n-1}(x)\ is\ Hermite\ polynomial$$

erf (x) $=\pi^{-1/2}\gamma\left(\frac{1}{2}, x^2\right)$, $where\ \gamma\left(\frac{1}{2}, x^2\right)\ could\ be\ view\ as\ incomplete\ gamma\ function$, it can therefore be expressed by the Maclaurin series in:

$$= \frac{2}{\sqrt{\pi}}\ \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(2n+1)} \quad \textbf{(10)}$$

By substituting for the error function as a Maclaurin series in (10), we obtain (11):

$$= (1+\lambda)\frac{\sqrt{2}}{\sigma\sqrt{\pi}}\int_{\mathbf{0}}^{\infty} \boldsymbol{x^r} e^{-\frac{x^2}{2\sigma^2}} \boldsymbol{dx} - \mathbf{2}\lambda\frac{\sqrt{2}}{\sigma\sqrt{\pi}}\int_{0}^{\infty} \boldsymbol{x^r}\frac{2}{\sqrt{\pi}}\ \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{n!(2n+1)} e^{-\frac{x^2}{2\sigma^2}} dx \quad \textbf{(11)}$$

$$= (1+\lambda)\frac{\sqrt{2}}{\sigma\sqrt{\pi}}\ \int_{\mathbf{0}}^{\infty} \boldsymbol{x^r} e^{-\frac{x^2}{2\sigma^2}} dx - \mathbf{2}\lambda\frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\ \int_{0}^{\infty} \boldsymbol{x^r}\, x^{2n+1} e^{-\frac{x^2}{2\sigma^2}}\, dx \quad \textbf{(12)}$$

$$= (1+\lambda)\frac{\sqrt{2}}{\sigma\sqrt{\pi}}\ \int_{\mathbf{0}}^{\infty} \boldsymbol{x^r} e^{-\frac{x^2}{2\sigma^2}} \boldsymbol{dx}\ -\ \mathbf{2}\lambda\frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\int_{0}^{\infty} x^{2nr+r}\, e^{-\frac{x^2}{2\sigma^2}} dx \quad \textbf{(13)}$$

$$\text{Let } y = \frac{x^2}{2\sigma^2}\ ;\ x = \sigma\sqrt{2y} \quad ; \frac{dy}{dx} = \frac{2x}{2\sigma^2}\ ;\ 2\sigma^2 dy = 2x dx\ ;\ \frac{\sigma^2 dy}{x} = dx$$

$$= (1+\lambda)\frac{\sqrt{2}}{\sigma\sqrt{\pi}}\int_{0}^{\infty} x^r\, e^{-y}\frac{\sigma^2 dy}{x} - \mathbf{2}\lambda\frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\ \int_{0}^{\infty} x^{2nr+r}\ e^{-y}\frac{\sigma^2 dy}{x} \quad \textbf{(14)}$$

$$= (1+\lambda)\frac{\sigma^2\sqrt{2}}{\sigma\sqrt{\pi}}\int_0^\infty x^{r-1}\,e^{-y}dy - \mathbf{2}\lambda\frac{2\sigma^2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\int_0^\infty x^{2nr+r-1}\,e^{-y}\,dy \quad \textbf{(15)}$$

$$= (1+\lambda)\frac{\sigma\sqrt{2}}{\sqrt{\pi}}\int_0^\infty\left(\sigma\sqrt{2y}\right)^{r-1}e^{-y}dy - \mathbf{2}\lambda\frac{2\sigma^2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\int_0^\infty\left(\sigma\sqrt{2y}\right)^{2nr+r-1}e^{-y}\,dy \quad \textbf{(16)}$$

$$=(1+\lambda)\frac{\sigma\sqrt{2}}{\sqrt{\pi}}\left(\sigma\sqrt{2}\right)^{r-1}\int_0^\infty y^{\frac{r}{2}-\frac{1}{2}}e^{-y}dy - \mathbf{2}\lambda\frac{2\sigma^2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\left(\sigma\sqrt{2}\right)^{2nr+r-1}\int_0^\infty y^{\frac{nr}{2}+\frac{r}{2}-\frac{1}{2}}\,e^{-y}dy \quad \textbf{(17)}$$

$$= (1+\lambda)\frac{\left(\sigma\sqrt{2}\right)^r}{\sqrt{\pi}}\int_0^\infty y^{\frac{r}{2}-\frac{1}{2}}e^{-y}dy - \mathbf{2}\lambda\frac{2\sigma^2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\left(\sigma\sqrt{2}\right)^{2nr+r-1}\int_0^\infty y^{\frac{nr}{2}+\frac{r}{2}-\frac{1}{2}}\,e^{-y}dy \quad \textbf{(18)}$$

$$\boldsymbol{Ex^r}=(1+\lambda)\frac{\left(\sigma\sqrt{2}\right)^r}{\sqrt{\pi}}\Gamma\left(\frac{r+1}{2}\right) - \mathbf{2}\lambda\frac{2\sigma^2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\left(\sigma\sqrt{2}\right)^{2nr+r-1}\Gamma\left(\frac{nr+r+1}{2}\right) \quad \textbf{(19)}$$

Equation (19) above represents the moment of Transmuted HalfNormal Distribution.

$$E(x) = (1+\lambda)\frac{\sigma\sqrt{2}}{\sqrt{\pi}}\Gamma\left(\frac{1+1}{2}\right) - \mathbf{2}\lambda\frac{2\sigma^2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\left(\sigma\sqrt{2}\right)^{2n+1-1}\Gamma\left(\frac{n+1+1}{2}\right)\ r=1; \quad \textbf{(20)}$$

$$E(x) = (1+\lambda)\frac{\sigma\sqrt{2}}{\sqrt{\pi}} - \mathbf{2}\lambda\frac{2\sigma^2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\left(\sigma\sqrt{2}\right)^{2n}\Gamma\left(\frac{n+2}{2}\right)\quad \mathrm{r}=1$$

$$\boldsymbol{Ex^2}=(1+\lambda)\frac{\left(\sigma\sqrt{2}\right)^2}{\sqrt{\pi}}\Gamma\left(\frac{2+1}{2}\right) - \mathbf{2}\lambda\frac{2\sigma^2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\left(\sigma\sqrt{2}\right)^{4n+2-1}\Gamma\left(\frac{2n+2+1}{2}\right)\quad r=2; \quad \textbf{(21)}$$

$$\boldsymbol{Ex^2}=(1+\lambda)\frac{\left(\sigma\sqrt{2}\right)^2}{\sqrt{\pi}}\frac{\sqrt{\pi}}{2} - \mathbf{2}\lambda\frac{2\sigma^2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\left(\sigma\sqrt{2}\right)^{4n+1}\Gamma\left(\frac{2n+3}{2}\right);r=2; \quad \textbf{(22)}$$

$$\boldsymbol{Ex^2}=(1+\lambda)\frac{\left(\sigma\sqrt{2}\right)^2}{2} - \mathbf{2}\lambda\frac{2\sigma^2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\left(\sigma\sqrt{2}\right)^{4n+1}\Gamma\left(\frac{2n+3}{2}\right);r=2 \quad \textbf{(23)}$$

$$\boldsymbol{Ex^3}=(1+\lambda)\frac{\left(\sigma\sqrt{2}\right)^3}{\sqrt{\pi}}\Gamma\left(\frac{3+1}{2}\right) - \mathbf{2}\lambda\frac{2\sigma^2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\left(\sigma\sqrt{2}\right)^{6n+3-1}\Gamma\left(\frac{3n+3+1}{2}\right)\ r=3; \quad \textbf{(24)}$$

$$\boldsymbol{Ex^3}=(1+\lambda)\frac{\left(\sigma\sqrt{2}\right)^3}{\sqrt{\pi}} - \mathbf{2}\lambda\frac{2\sigma^2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\left(\sigma\sqrt{2}\right)^{6n+2}\Gamma\left(\frac{3n+4}{2}\right)\ r=3 \quad \textbf{(25)}$$

$$Ex^4 = (1+\lambda)\frac{3(\sigma\sqrt{2})^4}{4} - 2\lambda\frac{2\sigma^2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\left(\sigma\sqrt{2}\right)^{8n+3}\Gamma\left(\frac{nr+5}{2}\right)\ r=4 \tag{26}$$

Moment about the Mean

$$\text{Variance }(\mu_2) = Ex^2 - \left(E(x)\right)^2$$

$$= (1+\lambda)\frac{(\sigma\sqrt{2})^2}{\sqrt{\pi}}\frac{\sqrt{\pi}}{2} - 2\lambda\frac{2\sigma^2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\left(\sigma\sqrt{2}\right)^{4n+1}\Gamma\left(\frac{2n+3}{2}\right) - \left[(1+\lambda)\frac{\sigma\sqrt{2}}{\sqrt{\pi}} - 2\lambda\frac{2\sigma^2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\left(\sigma\sqrt{2}\right)^{2n}\Gamma\left(\frac{n+2}{2}\right)\right]^2 \tag{27}$$

$$\mu_3 = E(x-\mu)^3$$

By applying Binomial Expansion, it gives:

$$= (1+\lambda)\frac{(\sigma\sqrt{2})^3}{\sqrt{\pi}} - 2\lambda\frac{2\sigma^2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\left(\sigma\sqrt{2}\right)^{6n+2}\Gamma\left(\frac{3n+4}{2}\right) - 3(1+\lambda)\frac{\sigma\sqrt{2}}{\sqrt{\pi}} - 2\lambda\frac{2\sigma^2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\left(\sigma\sqrt{2}\right)^{2n}\Gamma\left(\frac{n+2}{2}\right)\Bigg((1+\lambda)\frac{(\sigma\sqrt{2})^2}{2} - 2\lambda\frac{2\sigma^2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\left(\sigma\sqrt{2}\right)^{4n+1}\Gamma\left(\frac{2n+3}{2}\right) + 2\left((1+\lambda)\frac{\sigma\sqrt{2}}{\sqrt{\pi}}\Gamma\left(\frac{1+1}{2}\right) - 2\lambda\frac{2\sigma^2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\left(\sigma\sqrt{2}\right)^{2n+1-1}\Gamma\left(\frac{n+1+1}{2}\right)\right)^3\Bigg) \tag{28}$$

$$= E(x-\mu)^4 = \binom{4}{0}.x^0.(-\mu)^{4-0} + \binom{4}{1}.x^1.(-\mu)^{4-1} + \binom{4}{2}x^2.(-\mu)^{4-2} + \binom{4}{3}.x^3.(-\mu)^{4-3} + \binom{4}{4}.x^4.(-\mu)^{4-4}$$

$$\mu_4 = Ex^4 - 4\mu Ex^3 + 6\mu^2 Ex^2 - 3(Ex)^4$$

$$= (1+\lambda)\frac{3(\sigma\sqrt{2})^4}{4} - 2\lambda\frac{2\sigma^2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\left(\sigma\sqrt{2}\right)^{8n+3}\Gamma\left(\frac{nr+5}{2}\right) - 4\left((1+\lambda)\frac{\sigma\sqrt{2}}{\sqrt{\pi}} - 2\lambda\frac{2\sigma^2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\left(\sigma\sqrt{2}\right)^{2n}\Gamma\left(\frac{n+2}{2}\right)\right)(1+\lambda)\frac{(\sigma\sqrt{2})^3}{\sqrt{\pi}} - 2\lambda\frac{2\sigma^2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\left(\sigma\sqrt{2}\right)^{6n+2}\Gamma\left(\frac{3n+4}{2}\right) + 6\left((1+\lambda)\frac{\sigma\sqrt{2}}{\sqrt{\pi}} - 2\lambda\frac{2\sigma^2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\left(\sigma\sqrt{2}\right)^{2n}\Gamma\left(\frac{n+2}{2}\right)\right)^2\left((1+\lambda)\frac{(\sigma\sqrt{2})^2}{2} - 2\lambda\frac{2\sigma^2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\left(\sigma\sqrt{2}\right)^{4n+1}\Gamma\left(\frac{2n+3}{2}\right) - 3\left((1+\lambda)\frac{\sigma\sqrt{2}}{\sqrt{\pi}} - 2\lambda\frac{2\sigma^2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\left(\sigma\sqrt{2}\right)^{2n}\Gamma\left(\frac{n+2}{2}\right)\right)^4\right) \tag{29}$$

Skewness

This is obtained as:

$$\boldsymbol{\gamma_1(x)} = \frac{(\mu_3)^2}{(\mu_2)^3} = \frac{\left(\begin{array}{c} (1+\lambda)\frac{(\sigma\sqrt{2})^3}{\sqrt{\pi}} - \mathbf{2}\lambda\frac{2\sigma^2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}(\sigma\sqrt{2})^{6n+2}\,\Gamma\left(\frac{3n+4}{2}\right) - 3(1+\lambda)\frac{\sigma\sqrt{2}}{\sqrt{\pi}} - \mathbf{2}\lambda\frac{2\sigma^2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}(\sigma\sqrt{2})^{2n} \\ \Gamma\left(\frac{n+2}{2}\right)\left((1+\lambda)\frac{(\sigma\sqrt{2})^2}{2} - \mathbf{2}\lambda\frac{2\sigma^2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}(\sigma\sqrt{2})^{4n+1}\,\Gamma\left(\frac{2n+3}{2}\right) + 2\left((1+\lambda)\frac{\sigma\sqrt{2}}{\sqrt{\pi}}\Gamma\left(\frac{1+1}{2}\right) - \mathbf{2}\lambda\frac{2\sigma^2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}(\sigma\sqrt{2})^{2n+1-1}\,\Gamma\left(\frac{n+1+1}{2}\right)\right)^3\right) \end{array}\right)^{\mathbf{2}}}{\left((1+\lambda)\frac{(\sigma\sqrt{2})^2}{\sqrt{\pi}}\frac{\sqrt{\pi}}{2} - \mathbf{2}\lambda\frac{2\sigma^2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}(\sigma\sqrt{2})^{4n+1}\,\Gamma\left(\frac{2n+3}{2}\right) - \left[(1+\lambda)\frac{\sigma\sqrt{2}}{\sqrt{\pi}} - \mathbf{2}\lambda\frac{2\sigma^2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}(\sigma\sqrt{2})^{2n}\,\Gamma\left(\frac{n+2}{2}\right)\right]^2\right)^{\mathbf{3}}}$$

(30)

Kurtosis

This was derived as shown below:

$$\boldsymbol{\gamma_2(x)} = \frac{\mu_4}{(\mu_2)^2}$$

$$= \frac{\left(\begin{array}{c} (1+\lambda)\frac{3(\sigma\sqrt{2})^4}{4} - \mathbf{2}\lambda\frac{2\sigma^2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}(\sigma\sqrt{2})^{8n+3}\,\Gamma\left(\frac{nr+5}{2}\right) - 4\left((1+\lambda)\frac{\sigma\sqrt{2}}{\sqrt{\pi}} - \mathbf{2}\lambda\frac{2\sigma^2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}(\sigma\sqrt{2})^{2n}\,\Gamma\left(\frac{n+2}{2}\right)\right) \\ (1+\lambda)\frac{(\sigma\sqrt{2})^3}{\sqrt{\pi}} - \mathbf{2}\lambda\frac{2\sigma^2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}(\sigma\sqrt{2})^{6n+2}\,\Gamma\left(\frac{3n+4}{2}\right) + 6\left((1+\lambda)\frac{\sigma\sqrt{2}}{\sqrt{\pi}} - \mathbf{2}\lambda\frac{2\sigma^2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}(\sigma\sqrt{2})^{2n}\,\Gamma\left(\frac{n+2}{2}\right)\right)^2 \\ (1+\lambda)\frac{(\sigma\sqrt{2})^2}{2} - \mathbf{2}\lambda\frac{2\sigma^2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}(\sigma\sqrt{2})^{4n+1}\,\Gamma\left(\frac{2n+3}{2}\right) - 3\left((1+\lambda)\frac{\sigma\sqrt{2}}{\sqrt{\pi}} - \mathbf{2}\lambda\frac{2\sigma^2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}(\sigma\sqrt{2})^{2n}\,\Gamma\left(\frac{n+2}{2}\right)\right)^4 \end{array}\right)}{\left((1+\lambda)\frac{(\sigma\sqrt{2})^2}{\sqrt{\pi}}\frac{\sqrt{\pi}}{2} - \mathbf{2}\lambda\frac{2\sigma^2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}(\sigma\sqrt{2})^{4n+1}\,\Gamma\left(\frac{2n+3}{2}\right) - \left[(1+\lambda)\frac{\sigma\sqrt{2}}{\sqrt{\pi}} - \mathbf{2}\lambda\frac{2\sigma^2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}(\sigma\sqrt{2})^{2n}\,\Gamma\left(\frac{n+2}{2}\right)\right]^2\right)^{\mathbf{2}}}$$

Hazard Rate Function

$$= \boldsymbol{h}(\boldsymbol{x};\lambda,\sigma) = \frac{[\mathbf{1}+\lambda-\mathbf{2}\lambda\mathrm{G(x)}]\boldsymbol{g}(\boldsymbol{x})}{[\mathbf{1}-\boldsymbol{G}(\boldsymbol{x})][\mathbf{1}-\lambda\mathrm{G(x)}]} = \boldsymbol{h}_{\boldsymbol{G}}(\boldsymbol{x};\sigma)\,\frac{[\mathbf{1}+\lambda-\mathbf{2}\lambda\mathrm{G(x)}]}{\mathbf{1}-\lambda\mathrm{G(x)}}$$

where $\boldsymbol{h}_{\boldsymbol{G}}(\boldsymbol{x};\sigma)$ is the baseline (parent) distribution.

$$= \boldsymbol{h}(\boldsymbol{x};\lambda,\sigma) = \frac{\frac{\sqrt{2}}{\sigma\sqrt{\pi}}e^{-\frac{x^2}{2\sigma^2}}}{\mathbf{1}-\boldsymbol{erf}\left(\frac{x}{\sigma\sqrt{2}}\right)} \cdot \frac{\mathbf{1}+\lambda-\mathbf{2}\lambda\,.\mathrm{erf}\left(\frac{x}{\sigma\sqrt{2}}\right)}{\mathbf{1}-\lambda\mathrm{erf}\left(\frac{x}{\sigma\sqrt{2}}\right)} = \frac{\left[\mathbf{1}+\lambda-\mathbf{2}\lambda\mathrm{erf}\left(\frac{x}{\sigma\sqrt{2}}\right)\right]\frac{\sqrt{2}}{\sigma\sqrt{\pi}}e^{-\frac{x^2}{2\sigma^2}}}{\left[\mathbf{1}-\boldsymbol{erf}\left(\frac{x}{\sigma\sqrt{2}}\right)\right]\left[\mathbf{1}-\lambda\mathrm{erf}\left(\frac{x}{\sigma\sqrt{2}}\right)\right]}$$

(31)

$$= \frac{\frac{\sqrt{2}}{\sigma\sqrt{\pi}}e^{-\frac{x^2}{2\sigma^2}}+\lambda\frac{\sqrt{2}}{\sigma\sqrt{\pi}}e^{-\frac{x^2}{2\sigma^2}}-2\lambda\mathrm{erf}\left(\frac{x}{\sigma\sqrt{2}}\right)}{\left[1-\boldsymbol{erf}\left(\frac{x}{\sigma\sqrt{2}}\right)\right]\left[1-\lambda\mathrm{erf}\left(\frac{x}{\sigma\sqrt{2}}\right)\right]}$$

(32)

Order Statistics

According to Marcelo Bourguignon et al (2016), the order statistics for transmuted family can be obtained by (30).

$=f_{r.n}(x;\lambda,\sigma) = \frac{1}{B(r,n-r+1)}F(x)^{r-1}[1-F(x)]^{n-r}$ (30) which can be view (30) when defining it in term of transmuted family using the required baseline (parent) distribution.

$$=\frac{1}{B(r,n-r+1)}\sum_{j=0}^{n-r}(-1)^j\binom{n-j}{j}[(1+\lambda)G(x;\sigma)-\lambda(G(x;\sigma))^2]^{r-1+j}.[1+\lambda-2\lambda G(x;\sigma)]g(x)$$

By substituting the pdf and cdf of parent distribution we obtain the order statistics of transmuted HalfNormal distribution (THND) .

$$=\frac{1}{B(r,n-r+1)}\sum_{j=0}^{n-r}(-1)^j\binom{n-j}{j}\left[(1+\lambda)\boldsymbol{erf}\left(\frac{x}{\sigma\sqrt{2}}\right)-(\boldsymbol{erf}\left(\frac{x}{\sigma\sqrt{2}}\right))^2\right]^{r-1+j}\left[1+\lambda-2\lambda\boldsymbol{erf}\left(\frac{x}{\sigma\sqrt{2}}\right)\right]\frac{\sqrt{2}}{\sigma\sqrt{\pi}}e^{-\frac{x^2}{2\sigma^2}}$$

(33)

The k^th order moment of $X_{r,n}$ is displayed below:

$$=\int_0^\infty x^k\left[\left(1+\lambda G(x;\sigma)-\lambda\left(G(x)\right)^2\right)\right]^{r+j-1}[1+\lambda-2\lambda G(x;\sigma)]g(x;\sigma)$$

(34)

Equation (34) can be simplified by binomial expansion as (35):

$$=\frac{1}{B(r,n-r+1)}\sum_{j=0}^{n-r}(-1)^j\binom{n-r}{j}\ J$$ **(35)**

$J=\int_0^1\frac{1}{G(x;\sigma)}.(1+\lambda-2\lambda t)\ .\ [(1+\lambda)t-\lambda t^2]^{r+j-1}$, substitute for G(x; σ) in J to obtain the k^{th} order moment of Transmuted Half-Normal.

$J=\int_0^1\frac{1}{erf\left(\frac{x}{\sigma\sqrt{2}}\right)}.(1+\lambda-2\lambda t)\ .\ [(1+\lambda)t-\lambda t^2]^{r+j-1}$ Put for J in Equation (35).

$$=\frac{1}{B(r,n-r+1)}\sum_{j=0}^{n-r}(-1)^j\binom{n-r}{j}\int_0^1\frac{1}{erf\left(\frac{x}{\sigma\sqrt{2}}\right)}\cdot(1+\lambda-2\lambda t)\ .\ [(1+\lambda)t-\lambda t^2]^{r+j-1}$$

(36)

J can be evaluated using numerical integration.

Maximum Likelihood Method for THND

We consider the estimation of parameters of Transmuted family from samples by maximum likelihood. Let $x_1\ ,x_2,.\ .\ .x_n$ be observed values from this family with parameter$\theta=\lambda,\ \sigma$.

$$Lf_{THND}(x;\lambda,\sigma)\ =\ \frac{2^{\frac{n}{2}}}{\pi^{\frac{n}{2}}\sigma^n}e^{-\sum_{i=1}^{n}\frac{x^2}{2\sigma^2}}+\sum_{i=1}^{n}\left[1+\lambda-2\lambda erf\left(\frac{x}{\sigma\sqrt{2}}\right)\right] \quad \textbf{(37)}$$

$$lnLf_{THND}(x;\lambda,\sigma)=n\left(\frac{1}{2}ln2-\frac{1}{2}ln\pi-ln\sigma\right)+\ \sum_{i=1}^{n}ln\left[1+\lambda-2\lambda\boldsymbol{erf}\left(\frac{x}{\sigma\sqrt{2}}\right)\right] \quad \textbf{(38)}$$

$$\frac{\partial lnLf_{THND}(x;\lambda,\sigma)}{\partial\lambda}=\ \sum_{i=0}^{n}\left(\frac{1-2\boldsymbol{erf}\left(\frac{x}{\sigma\sqrt{2}}\right)}{1+\lambda-2\lambda\boldsymbol{erf}\left(\frac{x}{\sigma\sqrt{2}}\right)}\right) \quad \textbf{(39)}$$

$$\frac{\partial lnLf_{THND}(x;\lambda,\sigma)}{\partial\sigma}\ =\ -\frac{n}{\sigma}+\sum_{i=0}^{n}\left(\frac{\frac{2\lambda\sqrt{2}(\sigma^2-x^2)}{\sigma^4\sqrt{\pi}}e^{-\frac{x^2}{2\sigma^2}}}{1+\lambda-2\lambda\boldsymbol{erf}\left(\frac{x}{\sigma\sqrt{2}}\right)}\right) \quad \textbf{(40)}$$

To obtain the information matrix, we follow this procedure and we obtain $u_{\lambda\lambda}\ \ ,u_{\lambda\sigma}\ and\ \ u_{\sigma\sigma}$, we can form information matrix called J (θ) and the likelihood of the two distributions can be used to test for the goodness of fit to determine if transmuted distribution is superior to the baseline distribution based on the available data. These equations can be solved numerically using Newton-Raphson algorithm and the information matrix J (θ) is given by (36).

Fitting Transmuted HalfNormal to GLO-DATA Plan Data

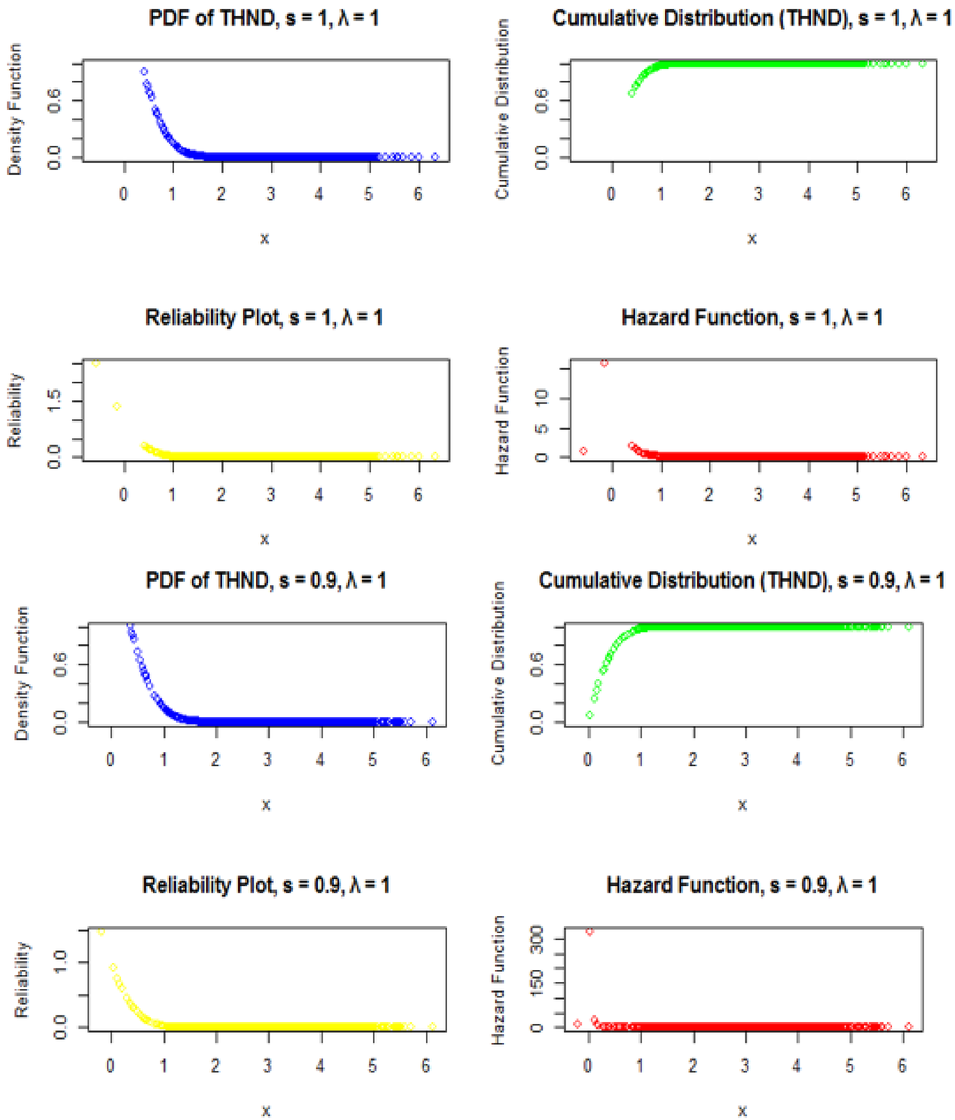

Figure 1: The Plot of THND at Different Values of Parameter (λ, σ).

From Figure 1 above, we can deduce that the distribution is heavily tailed.

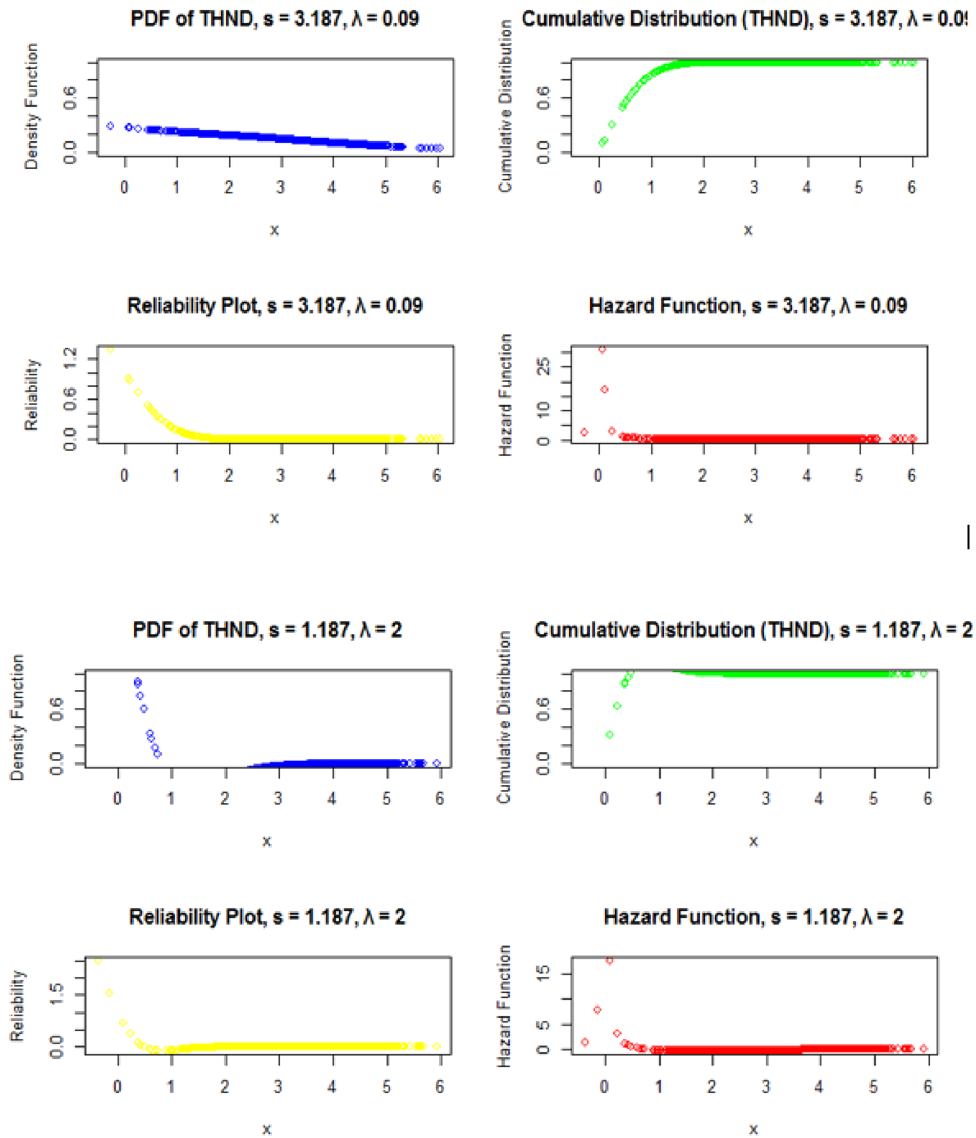

Figure 2: The Plot of THND at Different Values of Parameter (λ, σ).

From Figure 2 above, we can deduce that the distribution is heavily tailed and highly skewed at different parameter of the model, the transmuted parameter λ give the transmuted HalfNormal this flexibility and hence making it accurate when it come to modeling data whose distribution is heavily tailed and highly skewed.

ANALYSIS AND RESULTS

From Figure 3, we can deduce that the data is not normally distributed as it is heavily tailed, the THND can therefore be fitted to the data. Since this data shows the impression of the THND, then we can fit the hybrid model to the data.

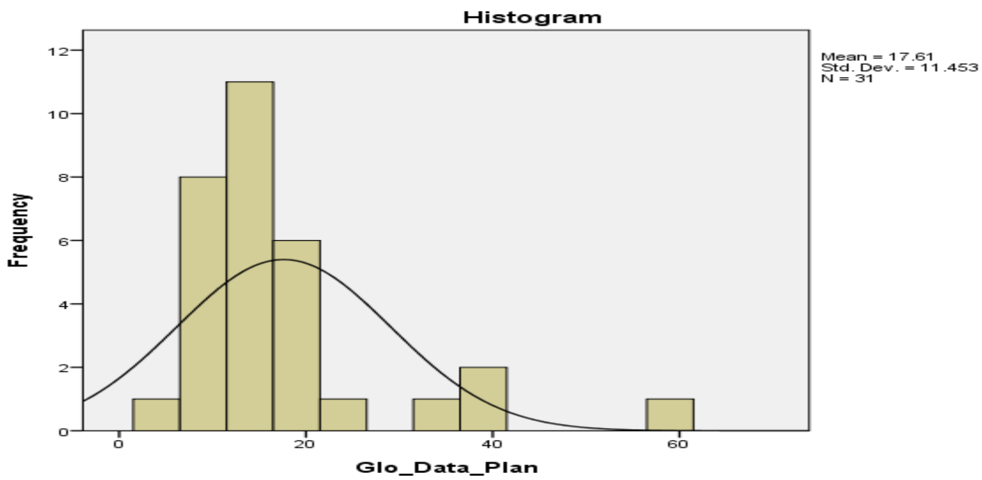

Figure 3: The Plot Showing the Distribution of the Data Generated from the GLO Data Plan.

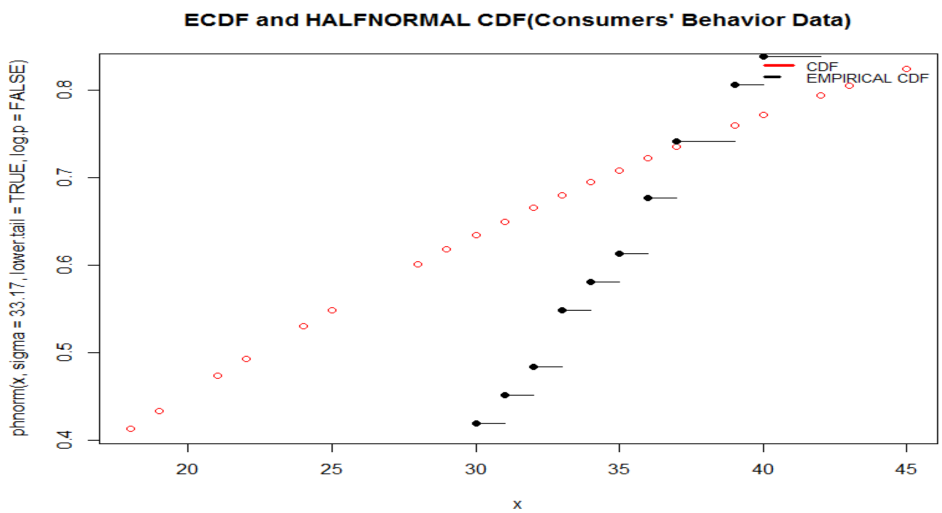

Figure 4: CDF and ECDF of HalfNormal Distribution when fitted to Buying Behavior Data.

Fitting THND to the Available Data

Table 1: Parameter Estimate.

Model	THND [Transmuted HalfNormal]		HND [HalfNormal]	
Parameter	**Estimate**	**Std. Error**	**Estimate**	**Std. Error**
σ	7.178e-08	9.406e-17	33.317	3.425
Λ	3.614e-01	4.007e-11	-	-
Comparison Criterion	**AIC =** -1.857456e+20		**AIC =**264.3871	
Log Likelihood	9.287281e-19		131.1936	
	H= $\begin{pmatrix} 9.215123e+31 & -3.228008e+26 \\ -3.228008e+26 & -5.078385e+20 \end{pmatrix}$ w = 5.006941e+19			

From the Figure 4 above, we can say that the HalfNormal Distribution does not fit the data reasonably well.

WALD TEST

Hypothesis

H_0: $\lambda = 0$, $\alpha = 0.05$ versus H_1: $\lambda \neq 0$

Decision Rule: Accept H_0

if $w < \chi_q{}^2$, *otherwise do not accept*

q is the number of parameters in the model or the number of rows of the variance-covariance matrix = $\chi_q{}^2$, at q = 2, is 0.103

Decision

Since the w (5.006941e+19) > $\chi_q{}^2$, then we have statistical reason not to accept H_0 and conclude that THND captures the data reasonably well as compared to parent distributions because of the additional parameter that controls the flexibility of the distribution.

CONCLUSION

The plot of the hybrid distribution that is the Transmuted HalfNormal distribution (THND) shows that the additional parameter (transmuted parameter) control the tail of the model by making it heavily tailed, from the monthly GLO Data Plan, since p value (0.5383) > α (0.05), then there is great statistical evidence that the data tested are not from a normally distributed population (the data are not normal). Then the THND can be fitted to the data because the distribution shows heavy tail due to addition of the transmuted parameter(λ). Therefore, THND can be used to capture non-normal data.

The Wald test carried out indicate that since the w(5.006941e+19) > $\chi_q{}^2(0.103)$, then we say that THND captures the data reasonably well as compared to parent distribution because of the additional parameter that controls the flexibility of the distribution.

The likelihood ratio test shows that $T\ (262.3872) > \chi^2{}_{(0.9,31)}$ (20.599), indicating that the transmuted HalfNormal Distribution (THND) fits the data reasonably well and is better than HND, this was established even when comparing the fitted GLO monthly data plan using the Akaike' information criterion.

By obtaining the transmuted version of probability distributions, we get the corresponding hybrid distribution with increased number of parameters which gives the newly compounded distribution more flexibility, consistency and stability.

After careful application of the newly compounded distribution to customer buying behavior of Monthly data consumption of the GLO Subscribers, we therefore conclude that the additional parameter (Transmuted Parameter) gives THND more flexibility over HND in modeling highly skewed and heavily tailed data.

REFERENCES

1. Akinsete, A., F. Famoye, and C. Lee. 2008. "The Beta-Pareto Distribution". *Statistics*. 42(6): 547-563.

2. Akomolafe, A.A. and A. Maradesa. 2017. "Beta-Halfnormal Distribution and Its Properties". *International Journal of Advance Research and Publication*. 1(4):17-22.

3. Akomolafe, A.A. 2016. "Consistency of a Mixture of the Different Distributions Approach to the Analysis of Buying Behaviour Data". *International Journal of Mathematical Research*. 5(2): 179-187.

4. Alzaatreh, A., F. Famoye, and C. Lee. 2013. "Weibull-Pareto Distribution and its Applications". *Communication in Statistics: Theory and Methods* 42: 1673-1691.

5. Alzaatreh, A., C. Lee, and F. Famoye. 2013. *A New Method for Generating Families of Continuous Distributions*. *METRON*. 71, 63-79.

6. Ashour, S. and M. Eltehiwy. 2013. "Transmuted Lomax Distribution". *American Journal of Applied Mathematics and Statistics*. 1(6): 121- 127.

7. Badmus, N.I., M. Ikegwu, and M. Emmanuel. 2013. "The Beta-Weighted Weibull Distribution: Some Properties and Application to Bladder Cancer Data". *Applied and Computational Mathematics.* 2:5.

8. Behairy, S.M., G. AL-Dayian, and A. El-Helbawy. 2016. "The Kumaraswamy-Burr Type III Distribution: Properties and Estimation". *British Journal of Mathematics and Computer Science*. 14(2): 1-21.

9. Bourguignon, M., R.B. Silva, L.M. Zea, and G.M. Cordeiro. 2013. "The Kumaraswamy-Pareto Distribution". *Journal of Statistical Theory and Applications*. 12(2): 129-144.

10. Bourguignon, M., R. Silva, and G. Cordeiro. 2014. "Weibull-G Family of Probability Distributions". *Data Science*. 12: 53-68.

11. Burr, I.W. 1942. "Cumulative Frequency Distribution". *Annals of Mathematical Statistics*. 13: 215-232.

12. Cristiane, R., M.C. Gauss, G.B. Clarice, G. Demeterio, and M.O. Edwin. 2011. "The Weibull Negative Binomial Distribution". *Advance Ana. Application in Statistics*. 22(1): 25-55.

13. El-Bassiouny, A.H., N.F. Abdo, and H.S. Shahen. 2015. "Exponential Lomax Distribution". *International Journal of Computer Application*. 121(13): 24-29.

14. Eugene, N., C. Lee, and F. Famoye. 2002. "Beta-Normal Distribution and its Applications". *Communications in Statistic: Theory and Methods*. 31(4): 497-512.

15. Kareema, A.K. and A.B. Mohammed. 2013. "Exponential Pareto Distribution". *Journal of Mathematical Theory and Modeling*. 3(5): 135-146.

16. Mahmoud, M.R. and R.M. Mandouh. 2013. "Transmuted Frechet". *Journal of Applied Science Research*. 9(1): 555-561.

17. Marcelo, B., G. Indranil, and G.M. Cordeiro. 2016. "General Results for the Transmuted Family of Distributions and New Models". Journal of Probability and Statistics. vol. 2016, Article ID 7208425, 12 pages, 2016. https://doi.org/10.1155/2016/7208425

18. Merovci, F. and L. Puka. 2014. "Transmuted Pareto Distribution". *Probstat Forum*. 7: 1-11.

19. Mohammad, D. and A. Muhammad. 2014. "On the Mixture of BurrXII and Weibull Distribution". *Journal of Statistics Application and Probability*. 3(2):251-267.

20. Nadarajah, S. and S. Kotz. 2006. "The Exponentiated Type Distribution". *Acta Applicandae Mathematica*. 92(2): 97-111.

21. Naradarajah, S. 2006. "The Exponential Gumbel Distribution with Climate Application". *Environmetrics*. 17(1): 13-23.

22. Oseghale, O.I. and A.A. Akomolafe. 2017. "Performance Rating of the Kumaraswamy Transmuted Weibull Distribution: An Analytic Approach". *American Journal of Mathematics and Statistics*. 7(3): 125-135.

APPENDIX

The data used in this project was collected using the standard wholesale outlet (Business Centre) where the Global Communication data is sold, and it depicts the data consumption of people that use GLO network for internet browsing
35, 39, 28, 19, 31, 42, 37, 25, 43, 30, 34, 37, 21, 29, 33, 36, 28, 29, 18, 22, 33, 42, 29, 45, 36, 24, 29, 32, 40, 39, 43, 42, 29, 28, 44, 41.

The data is the number of the retail agent that came to purchase GLO data bundles in some selected locations in Ibadan and Lagos for twelve consecutive weeks. This data was based on daily data consumption for the month of January 2019.

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