Extended Three-Step Iterative Method for Solving System of Nonlinear Equations

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ABSTRACT

In this paper, we present an extended three-step iterative method for solving system of nonlinear equations. After proving the convergence of the proposed algorithm, numerical experiments show that the method can compete with Newton's method in terms of order of convergence and number of iterations.

(Keywords: nonlinear equations, iterative method, convergence)

INTRODUCTION

Recently, numerous scholars developed several iterative method to solve nonlinear equations [1], [3], [5], [6], [12]. [1] developed a three-step iterative method with third-order of convergence for solving nonlinear equations using a different type of decomposition from [4], the method is very simple as compared with Adomian decomposition method and involves only the first derivative. [3] presented the correct mathematical development of the Three-step iterative methods for nonlinear equations developed by [1]. In favor of their claim, they proved that the computational order of convergence is not three rather four, and also provide the generalization of their method. [2] presented a third-order Newton-type method to solve system of nonlinear equations.

In this paper, we present the mathematical development of the three-step iterative method to solve $f(x)=0$ as introduced by [3], we extend the method to $n-$dimensional case and analyze the performance of the developed method.

Development of the Method

Let α be a simple root of nonlinear equation $f(\alpha)=0$, and γ be an initial guess in the vicinity of α. The Taylors series expansion of f around γ gives:

$$f(x)=0 \quad \textbf{(1)}$$

$$f(\gamma)+f'(\gamma)(x-\gamma)+g(x)=0, \quad \textbf{(2)}$$

where,

$$g(x)=f(x)-f(\gamma)-f'(\gamma)(x-\gamma) \quad \textbf{(3)}$$

dividing (2) by $f'(\gamma)$ they got:

$$x=c+N(x) \quad \textbf{(4)}$$

where,

$$c=\gamma-\frac{f(\gamma)}{f'(\gamma)} \quad \textbf{(5)}$$

$$N(x)=-\frac{g(x)}{f'(\gamma)} \quad \textbf{(6)}$$

Note that if:

$$f(x_0)=g(x_0)$$

Then (4) implies that:

$$x_0 = \gamma - \frac{f(\gamma)}{f'(\gamma)} = c \qquad \textbf{(7)}$$

and $N(x_0)$ is a nonlinear function.

Using the relations above, they now construct the three-step iterative method with the same decomposition used by [4] and using this decomposition, x is expressed:

$$x = \sum_{i=0}^{\infty} x_i \qquad \textbf{(8)}$$

Substituting (8) in (4), they obtained:

$$\sum_{i=0}^{\infty} x_i = c + N\left(\sum_{i=0}^{\infty} x_i\right) \qquad \textbf{(9)}$$

Hence the nonlinear function $N\left(\sum_{i=0}^{n} x_i\right)$ can be decomposed as:

$$N\left(\sum_{i=0}^{n} x_i\right) = N(x_0) + \sum_{i=1}^{\infty}\left[N\left(\sum_{j=0}^{i} x_j\right) - N\left(\sum_{j=0}^{i-1} x_j\right)\right] \qquad \textbf{(10)}$$

Substituting (10) in (9), they have:

$$\sum_{i=0}^{\infty} x_i = c + N(x_0) + \sum_{i=1}^{\infty}\left[N\left(\sum_{j=0}^{i} x_j\right) - N\left(\sum_{j=0}^{i-1} x_j\right)\right] \qquad \textbf{(11)}$$

By equating terms on both side of (11), it follows that:

$$\left.\begin{aligned}
x_0 &= c = \gamma - \frac{f(\gamma)}{f'(\gamma)} \\
x_1 &= N(x_0) = \frac{g(x)}{f'(\gamma)} \\
x_2 &= N(x_0 + x_1) - N(x_0) \\
&\vdots \\
x_{n+1} &= N(x_0 + x_1 + \cdots + x_n) - N(x_0 + x_1 + \cdots + x_{n-1}),\ \mathrm{n} = 2,3,\ldots
\end{aligned}\right\} \qquad \textbf{(12)}$$

In order to get the iterative scheme, they evaluated $N(x_0)$, and $N(x_0 + x_1)$ to obtained:

$$\left.\begin{aligned}
x_0 &= \gamma - \frac{f(\gamma)}{f'(\gamma)} \\
x_1 &= -\frac{f(x_0)}{f'(\gamma)} \\
x_2 &= -\frac{f(x_0 + x_1)}{f'(\gamma)}
\end{aligned}\right\} \qquad \textbf{(13)}$$

whereby, they approximated $x = x_0 + x_1 + x_2$, and got:

$$\left.\begin{aligned}
x_0 &= \gamma - \frac{f(\gamma)}{f'(\gamma)} \\
x_0 + x_1 &= x_0 - \frac{f(x_0)}{f'(\gamma)} \\
x = x_0 + x_1 + x_2 &= x_0 + x_1 - \frac{f(x_0 + x_1)}{f'(\gamma)}
\end{aligned}\right\} \qquad \textbf{(14)}$$

Hence, (14) suggested the following three-step iterative method:

$$\left.\begin{aligned}
y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\
z_n &= y_n - \frac{f(y_n)}{f'(x_n)} \\
x_{n+1} &= z_n - \frac{f(z_n)}{f'(x_n)}
\end{aligned}\right\} \qquad \textbf{(15)}$$

Development of the Proposed n-Dimensional Case

Consider the general form of nonlinear system of equation:

$$F(X) = 0 \qquad \textbf{(16)}$$

where $F=\left(f_1,f_2,\ldots,f_n\right)^T$ is a nonlinear mapping from $\Re^n$ to $\Re^n$, $X=\left(x_1,x_2,\ldots,x_n\right)^T\in\Re^n,\ n\in\mathrm{N}$. Let x^* be a solution of the nonlinear system (16) and x^0 be an initial guess in the neighborhood of x^*.We rewrite equations (14) and (15) in n-dimensional case as:

$$\left.\begin{aligned} x_0&=x^0-\left[F'\left(x^0\right)\right]^{-1}F\left(x^0\right)\\ x_0+x_1&=x_0-\left[F'\left(x^0\right)\right]^{-1}F\left(x_0\right)\\ x_0+x_1+x_2&=x_0+x_1-\left[F'\left(x^0\right)\right]^{-1}F\left(x_0+x_1\right)\end{aligned}\right\}\quad \textbf{(17)}$$

let $x_0=y_n$, $x_0+x_1=z_n$, $x_0+x_1+x_2=x_{n+1}$ and $x^0=x_n$ in (17), then the three-step iterative scheme to solve system of nonlinear equations is obtained as:

$$\left.\begin{aligned} y_n&=x_n-\left[F'\left(x_n\right)\right]^{-1}F\left(x_n\right)\\ z_n&=y_n-\left[F'\left(x_n\right)\right]^{-1}F\left(y_n\right)\\ x_{n+1}&=z_n-\left[F'\left(x_n\right)\right]^{-1}F\left(z_n\right)\end{aligned}\right\}\quad \textbf{(18)}$$

Iteration scheme (18) refers to as extended three-step iterative method to solve system of nonlinear equations.

Convergence Analysis of the Proposed Method

Theorem 1: [2]

Let $F:\Omega\subseteq\Re^n\to\Re^n$ is k-time Frechet differentiable in a convex set Ω containing the root x^* of $F(X)=0$. The extended three-step iterative method (18) converges quadratically.

Proof.

The Taylor series expansion of $F(x)$ about (x_n) is:

$$F(x)=F(x_n)+\frac{1}{1!}F'(x_n)(x-x_n)+\cdots+\frac{1}{k!}F^k(x_n)(x-x_n)^k+\cdots\quad \textbf{(19)}$$

setting $x=x^*$, (19) becomes:

$$F(x^*)=F(x_n)+\frac{1}{1!}F'(x_n)(x^*-x_n)+\cdots+\frac{1}{k!}F^k(x_n)(x^*-x_n)^k+\cdots\quad \textbf{(20)}$$

But $F(x^*)=0$, hence (20) can be rewritten as:

$$F(x_n)=F'(x_n)(x_n-x^*)+\cdots(-1)^k\frac{1}{k!}F^k(x_n)(x_n-x^*)^k+\cdots\quad \textbf{(21)}$$

The error at the *nth*-step is define as $e_n=x_n-x^*$, hence (21) can be rewritten as:

$$F(x_n)=F'(x_n)e_n-\frac{1}{2!}F''(x_n)e_n^2+\cdots(-1)^k\frac{1}{k!}F^k(x_n)e_n^k+\cdots\textbf{(22)}$$

For $k=2$ (22) can be written as:

$$F(x_n)=F'(x_n)e_n-\frac{1}{2!}F''(x_n)e_n^2+\mathrm{O}(\|e_n\|^3)\quad \textbf{(23)}$$

multiplying (23) by $F'(x_n)^{-1}$ from left we have:

$$F'(x_n)^{-1}F(x_n)=F'(x_n)^{-1}F'(x_n)e_n-\frac{1}{2!}F'(x_n)^{-1}F''(x_n)e_n^2+\mathrm{O}(\|e_n\|^3)$$

which implies:

$$F'(x_n)^{-1}F(x_n)=e_n-\frac{1}{2!}F'(x_n)^{-1}F''(x_n)e_n^2+\mathrm{O}(\|e_n\|^3)\quad \textbf{(24)}$$

Subtracting x^* from the first equation of (18) yields:

$$y_n-x^*=x_n-x^*-F'\left(x_n\right)^{-1}F\left(x_n\right)$$

Substituting (24) in the above equation gives:

$$y_n-x^*=x_n-x^*-\left[e_n-\frac{1}{2!}F'(x_n)^{-1}F''(x_n)e_n^2+\mathrm{O}(\|e_n\|^3)\right]\quad \textbf{(25)}$$

since $x_n-x^*=e_n$ and we let:

$$c_k = \frac{(-1)^k}{k!} F'(x_n)^{-1} F^k(x_n)$$, hence

$$y_n - x^* = c_2 e_n^2 + \mathrm{O}(\|e_n\|^3) \quad \textbf{(26)}$$

now Taylor series expansion of $F(y_n)$ about x^* is:

$$F(y_n) = F'(x^*) c_2 e_n^2 + \mathrm{O}\left(\|e_n\|^3\right) \quad \textbf{(27)}$$

Subtracting x^* from the second step of (18) gives:

$$z_n - x^* = y_n - x^* - \left[F'(x_n)\right]^{-1} F(y_n) \quad \textbf{(28)}$$

which upon substitution of (26) & (27) in (28) yields:

$$z_n - x^* = c_2 e_n^2 + \mathrm{O}(\|e_n\|^3) - \left[F'(x_n)\right]^{-1} F'(x^*)\left[c_2 e_n^2 + \mathrm{O}\left(\|e_n\|^3\right)\right] \quad \textbf{(29)}$$

The Taylor series expansion of $F(z_n)$ about x^* is:

$$F(z_n) = F'(x^*) c_2 e_n^2 - \left[F'(x_n)\right]^{-1} F'(x^*)^2 c_2 e_n^2 + \mathrm{O}\left(\|e_n\|^3\right) \quad \textbf{(30)}$$

Subtracting x^* from the third step of (18) yields:

$$x_{n+1} - x^* = z_n - x^* - \left[F'(x_n)\right]^{-1} F(z_n) \quad \textbf{(31)}$$

Substituting (29) & (30) in (31) yields:

$$x_{n+1} - x^* = c_2 e_n^2 - \left[F'(x_n)\right]^{-1} F'(x^*) c_2 e_n^2 - \left[F'(x_n)\right]^{-1}\left[F'(x^*) c_2 e_n^2 - \left[F'(x_n)\right]^{-1}\left[F'(x^*)\right]^2 c_2 e_n^2\right] + \mathrm{O}\left(\|e_n\|^3\right)$$

Collecting like terms result to:

$$x_{n+1} - x^* = \left[c_2 - 2c_2\left[F'(x_n)\right]^{-1} F'(x^*) + c_2\left(\left[F'(x_n)\right]^{-1}\right)^2\left[F'(x^*)\right]^2\right] e_n^2 + \mathrm{O}\left(\|e_n\|^3\right) \quad \textbf{(32)}$$

Let:

$$A_n = c_2 - 2c_2\left[F'(x_n)\right]^{-1} F'(x^*) + c_2\left(\left[F'(x_n)\right]^{-1}\right)^2\left[F'(x^*)\right]^2$$

to obtain:

$$x_{n+1} - x^* = A_n e_n^2 + \mathrm{O}\left(\|e_n\|^3\right) \quad \textbf{(33)}$$

Taking the norm of (33) gives:

$$\|x_{n+1} - x^*\| \le \|A_n\| \|e_n^2\| + \mathrm{O}\left(\|e_n\|^3\right) \quad \textbf{(34)}$$

If $\|A_n\| = A$, then (34) becomes:

$$\|x_{n+1} - x^*\| \le A \|e_n^2\| + \mathrm{O}\left(\|e_n\|^3\right) \quad \textbf{(35)}$$

which means the extended three step iterative method (18) has quadratic order convergence.

Numerical Examples

We consider the following problems to test the performance of the developed ETSM algorithms in comparison with Classical Newton Method (CNM) using some benchmark problems from the literature. The following notation are used in the tables below:

N - dimension of the problem,

NI - number of iteration,

CPU time (in seconds) - Central processing unit time

CNM - Classical Newton's Method,

ETSM - Extended Three-step Method.

Problem 1:

$$f_i(X) = e^{x_i} - 1$$

$$i = 1, 2, \ldots, n \text{ and } X_0 = (0.5, 0.5, \ldots, 0.5)^T$$

Source: [7]

Problem 2:

$$f_i(x) = x_i^2 - 1$$

$i = 1, 2, \ldots, n$ *and* $x_0 = (0.5, 0.5, \ldots, 0.5)^T$
Source: [9]

Problem 3:

$$F_i(x) = (1 - x_i^2) + x_i(1 + x_i x_{n-2} x_{n-1} x_n) - 2$$

$i = 1, 2, \ldots, n$ *and* $x_0 = (0.5, 0.5, \ldots, 0.5)^T$
Source: [10]

Problem 4:

$$f_i = (0.5 - x_i)^2 + x_{n+1-i}^2 - 0.2x_i - 1$$

$i = 1, 2, \ldots, n-1$ and $x_0 = (0.5, 0.5, \ldots, 0.5)^T$
Source: [8]

Problem 5:

$$f_i = x_i - 0.1x_{i+1}^2$$

$i = 1, 2, \ldots, n$ and $x_0 = (0.5, 0.5, \ldots, 0.5)^T$
Source: [11]

Table 1: Numerical Results of CNM and ETSM Methods on Problems 1-5.

Prob	N	CNM NI	CPU time	ETSM NI	CPU time
1	10	3	0.0000	3	0.0000
	100	3	0.0781	3	0.2344
	1000	3	8.8125	3	22.8125
2	10	4	0.0000	4	0.0156
	100	4	0.0938	4	0.2813
	1000	4	11.5938	4	30.3281
3	10	9	0.0000	9	0.0313
	100	9	0.2500	9	0.7344
	1000	9	23.6406	9	65.6719
4	10	5	0.0000	5	0.0156
	100	5	0.1406	5	0.4063
	1000	5	14.0469	5	38.2815
5	10	2	0.0000	2	0.0156
	100	2	0.0625	2	0.1406
	1000	2	6.7813	2	15.6406

DISCUSSION OF RESULTS

The numerical results presented in Table 1 indicate that ETSM can compete with classical Newton's method in terms of number of iterations, although it has higher CPU time which was due the fact that ETSM uses Newton's method in all the three-steps. As we can see from problem 1 through 5, CNM and ETSM have the same number of iterations for all the dimensions considered, that is $N = 10,\ 100\ \&\ 1{,}000$

CONCLUSION

In this paper, we extend the three-step iterative scheme in Malik *et al* [3] to $n-$dimensional case. The new method solves system of nonlinear equations and converges to the solution with a quadratic rate. Numerical results show that the method can compete with Newton's method in terms of number of iterations.

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