A Two-Step Hybrid Nonlinear Multistep Method for the Solution of Stiff and Singular First Order Initial Value Problems

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ABSTRACT

In this paper, the development of two-step hybrid non-linear method for the solution of stiff and singular first order initial value problems of ordinary differential equations using Pade approximation is discussed. The interpolation and collocation method was adopt in the development of the continuous nonlinear method (scheme). Stability analysis shows that the method is consistent, zero stable and therefore convergent, numerical examples show that the method compete favorably with existing methods.

(Keywords: rational approximate solution, interpolation, collocation, stiff problems, singular problems, convergent)

INTRODUCTION

Differential equations play an important role in the modelling of physical problems arising from almost every discipline of study such as economics, medicine, psychology, operation research, space technology, and even in areas such as biology, and astronomy. Interestingly, differential equations arising from the modelling of such physical phenomena often are very difficult or impossible to solve analytically. Hence, the need for the development of numerical methods to obtain approximate solutions becomes inevitable, (Kamoh, Gyemang, Soomiyol, 2017).

The techniques for solving DEs based on numerical approximation were developed before programmable computers existed. During World War II, it was common to find rooms of people (usually women) working on mechanical calculators to numerically solve systems of DEs for military calculations. Before programmable computers, it was also common to exploit analogies to electrical systems to design analogue computers to study numerical, thermal or chemical systems. As programmable computers have increased in speed and decreased in cost. Increasingly complex systems of differential equations can be solved with simple programs written to run on a common computer (Storey, 2004).

An ordinary differential equation, (ODE) is in the form:

$$y^{(m)} = f\left(x, y, ..., y^{(m-1)}\right), y(a) = y_0, y^{(1)}(a) = y_1, ..., y^{(m-1)}(a) = y_{m-1} \quad \textbf{(1)}$$

where $f : \mathrm{R} \times \mathrm{R}^m \to \mathrm{R}^m, y \in \mathrm{R}^m$.

f is assumed to be piecewise continuous and differentiable and also it is assumed to satisfy the Lipchitz condition which guarantees the existence and uniqueness theorem. Conventional numerical integrators (i.e., the Rung Kutta processes and the linear multistep formulas) whose derivation is based on polynomial approximation perform very poorly in the solution of problems possessing singularities. Methods developed using rational approximate solution has been reported to be effective for problems whose solution are stiff or possess singularities (Fatunla, 1980).

Egbako and Adeboye (2012) presented a one-step, six-order method for treating stiff differential equations using Pade approximation of the form:

$$R(x)=\frac{p(x)}{q(x)} \quad (2)$$

where

$$p(x)=\sum_{i=0}^{m} a_i x^i \text{ and } q(x)=1+\sum_{i=0}^{m} b_i x^i$$

Dalquist's model test equation was used to analyse its basic properties. The results found shows that the method is consistent and convergent. Numerical results and comparative analysis with some methods was also done and it shows that the method is very efficient and more accurate.

Ahmad, Yaacob and Murid (2005) introduced and developed explicit methods to solve stiff problems, they considered the stability property and the numerical results shows that their method was A-stable.

Odekunle, Oye, Adee, and Ademuliyi (2004) developed a class of inverse Runge-Kutta scheme for the numerical integration of singular problems, using the approximate solution of the form:

$$\sum_{i=0}^{k} \alpha_i e^{Ah(k-1)} y_{n+1} = h\sum_{i=0}^{k} \phi_{ki}(Ah) g_{n+1} \quad (3)$$

The method was found to be L-stable and performed efficiently when applied to problems involving singularities.

Abukhaled, Khuri, and Sayfy (2011) developed two numerical schemes for finding approximate solution of singular two-point boundary value problems, l'Hopital rule was used to remove the singularity due to the boundary condition at the initial point.

Hiroaki and Kensuke (2014), examined some numerical test of Padé approximation of the form

$$f^{(N)}(z) \equiv \frac{P_L(z)}{Q_M(z)} \equiv f^{[\frac{L}{M}]}(z) \quad (4)$$

where $P_L(z)$ is a polynomial of degree less than or equal to L and $Q_M(z)$ is a polynomial of degree less than or equal to M. For some typical functions with singularities such as simple pole, essential singularity, brunch cut, and natural boundary. It was shown that the simple pole and the essential singularity can be characterized by the poles of the Padé approximation.

Okosun and Ademuliyi (2007) described a three-step method for the numerical solution of ordinary differential equations with singularities using the approximate solution of the form:

$$y_{n+k} = \frac{y_n}{1+\sum_{i=1}^{k} b_j x_n^j} \quad (5)$$

and $k=2$. The scheme was based on rational functions approximation technique and its development and analysis was based on power series expansions and Dalhquist stability test method. The scheme is convergent and A-stable. Numerical results show that the scheme is accurate, effective and efficient.

Fatunla (1982) presented a paper on nonlinear multistep method (NLMM) for IVPs. The scholar developed a method by approximating the theoretical solution by:

$$f_k(x) = \frac{A}{1+\sum_{r=1}^{k} a_r x^r} \quad (6)$$

some k-step, k-th order nonlinear multistep methods was developed for the solution of both stiff and singular IVPs. The scheme was stable and convergent.

Motsa and Sibanda (2012) presented a paper titled numerical approach for the solution of nonlinear singular boundary value problems arising in physiology. The approach is based on application of the successive linearization method (SLM), the method was found to give accurate results comparable to results in the literature found using existing numerical methods.

Umar, et al., (2019) discusses the formulation and implementation of non-linear method for the solution of singular initial value problem using collocation and interpolation of an approximate solution of the form (2). The method developed is consistent and convergent. Numerical examples show that the method is suitable for singular problems whose initial values are not zero.

MATHEMATICAL BACKGROUND

Consider an approximation solution of the form (2) where a_i's and b_i's are constants to be determined.

The first derivative of (2) gives:

$$R'(x)=\sum_{i=1}^{m} i a_i x^{i-1}-\sum_{i=1}^{n}(R(x) i b_i x^{i-1}+R'(x) b_i x^i) \tag{7}$$

Interpolating (2) at $x_{n+j},\ j=0,1,2,...,r$ and collocating (7) at $x_{n+j},\ j=0,1,2,...,s,$ gives a system of non-linear equations of the form:

$$U=AX \tag{8}$$

Where:

$$A=\begin{bmatrix} a_0 & a_1 & \cdots & a_m & b_1 & \cdots & b_n \end{bmatrix}^T$$

$$U=\begin{bmatrix} y_n & y_{n+1} & \cdots & y_{n+r} & f_{n+1} & \cdots & f_{n+s} \end{bmatrix}^T$$

$$X=\begin{bmatrix} 1 & \cdots & x_n^m & x_n y_n & \cdots & x_n^m y_n \\ 1 & \cdots & x_{n+1}^m & x_{n+1} y_{n+1} & \cdots & x_{n+1}^m y_{n+1} \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ 1 & \cdots & x_{n+r}^m & x_{n+r} y_{n+r} & \cdots & x_{n+r}^m y_{n+r} \\ 0 & \cdots & m x_n^{m-1} & -\left(x_n y_n' + y_n\right) & \cdots & -\left(x_n^m y_n' + m x_n^{m-1} y_n\right) \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & m x_{n+s}^{m-1} & -\left(x_{n+s} y_{n+s}' + y_{n+s}\right) & \cdots & -\left(x_{n+s}^m y_{n+s}' + m x_{n+s}^{m-1} y_{n+s}\right) \end{bmatrix}$$

imposing the following conditions on (2):

$$R(x_{n+j})=y_{n+j},\ j=0,1,2,...,r$$
$$R'(x_{n+j})=f_{n+j},\ j=0,1,2,...,s$$

In Umar, et al., (2019), points of evaluation were; u=1/2, v=1 w=2 and u=1, v=3/2, w=2. but in this work, the unknown constants in (8) were determined using Gaussian elimination and then substituting the results into (2) and evaluating the resultant at case I: $u=\frac{1}{4}, v=\frac{1}{2}, \omega=2$ and case II: $u=\frac{5}{4}, v=\frac{3}{2}, \omega=2$, gives the continuous non-linear method (CNLM) for the two cases in the form:

$$y_{n+2} = \frac{\left(\begin{array}{c} -32y_n^2 y_{n+2} + 56y_n^2 y_{n+\frac{1}{2}} - 24y_n^2 y_{n+\frac{1}{4}} - 32hy_n^2 f_{n+2} + 14hy_n^2 f_{n+\frac{1}{2}} \\ -3hy_n^2 f_{n+\frac{1}{4}} - 24y_n y_{n+2} y_{n+\frac{1}{2}} + 56y_n y_{n+2} y_{n+\frac{1}{4}} - 32y_n y_{n+\frac{1}{2}} y_{n+\frac{1}{4}} \\ +72h^3 f_n f_{n+2} f_{n+\frac{1}{2}} - 28h^3 f_n f_{n+2} f_{n+\frac{1}{4}} + 96h^2 f_n f_{n+2} y_{n+\frac{1}{2}} \\ +84h^2 f_n y_{n+2} f_{n+\frac{1}{2}} - 192h^2 y_n f_{n+2} f_{n+\frac{1}{2}} - 96h^2 f_n f_{n+2} y_{n+\frac{1}{4}} \\ -30h^2 f_n y_{n+2} f_{n+\frac{1}{4}} + 140h^2 y_n f_{n+2} f_{n+\frac{1}{4}} - 2h^3 f_n f_{n+\frac{1}{2}} f_{n+\frac{1}{4}} \\ -12h^2 f_n f_{n+\frac{1}{4}} y_{n+\frac{1}{2}} + 10h^2 y_n f_{n+\frac{1}{2}} f_{n+\frac{1}{4}} - 32hf_{n+2} y_{n+\frac{1}{2}} y_{n+\frac{1}{4}} \\ +224hy_{n+2} f_{n+\frac{1}{2}} y_{n+\frac{1}{4}} - 192hy_{n+2} f_{n+\frac{1}{4}} y_{n+\frac{1}{2}} + 144hf_n y_{n+2} y_{n+\frac{1}{2}} \\ +32hy_n f_{n+2} y_{n+\frac{1}{2}} - 206hy_n y_{n+2} f_{n+\frac{1}{2}} - 112hf_n y_{n+2} y_{n+\frac{1}{4}} \\ +32hy_n f_{n+2} y_{n+\frac{1}{4}} + 143hy_n y_{n+2} f_{n+\frac{1}{4}} - 32hf_n y_{n+\frac{1}{2}} y_{n+\frac{1}{4}} \\ -32hy_n f_{n+\frac{1}{2}} y_{n+\frac{1}{4}} + 52hy_n f_{n+\frac{1}{4}} y_{n+\frac{1}{2}} - 42h^3 f_{n+2} f_{n+\frac{1}{2}} f_{n+\frac{1}{4}} \\ +192h^2 f_{n+2} f_{n+\frac{1}{2}} y_{n+\frac{1}{4}} - 140h^2 f_{n+2} f_{n+\frac{1}{4}} y_{n+\frac{1}{2}} - 52h^2 y_{n+2} f_{n+\frac{1}{2}} f_{n+\frac{1}{4}} \end{array}\right)}{\left(\begin{array}{c} -32y_n y_{n+2} + 56y_n y_{n+\frac{1}{2}} - 24y_n y_{n+\frac{1}{4}} - 24y_{n+2} y_{n+\frac{1}{2}} \\ +56y_{n+2} y_{n+\frac{1}{4}} - 32y_{n+\frac{1}{2}} y_{n+\frac{1}{4}} + 32hf_n y_{n+2} + 112hf_n y_{n+\frac{1}{2}} \\ -210hy_n f_{n+\frac{1}{2}} - 144hf_n y_{n+\frac{1}{4}} + 189hy_n f_{n+\frac{1}{4}} - 42h^2 f_{n+\frac{1}{2}} f_{n+\frac{1}{4}} \\ +84h^2 f_n f_{n+\frac{1}{2}} - 42h^2 f_n f_{n+\frac{1}{4}} + 18hy_{n+2} f_{n+\frac{1}{2}} - 49hy_{n+2} f_{n+\frac{1}{4}} \\ +192hf_{n+\frac{1}{2}} y_{n+\frac{1}{4}} - 140hf_{n+\frac{1}{4}} y_{n+\frac{1}{2}} \end{array}\right)} \qquad 9$$

and

$$y_{n+2}=\frac{\left(\begin{array}{c}32y_n^2y_{n+2}-72y_n^2y_{n+\frac{3}{2}}+40y_n^2y_{n+\frac{5}{4}}+32hy_n^2f_{n+2}-54hy_n^2f_{n+\frac{3}{2}}\\+25hy_n^2f_{n+\frac{5}{4}}+40y_ny_{n+2}y_{n+\frac{3}{2}}-72y_ny_{n+2}y_{n+\frac{5}{4}}+32y_ny_{n+\frac{3}{2}}y_{n+\frac{5}{4}}\\+24h^3f_nf_{n+2}f_{n+\frac{3}{2}}-60h^3f_nf_{n+2}f_{n+\frac{5}{4}}-32h^2f_nf_{n+2}y_{n+\frac{3}{2}}\\+60h^2f_ny_{n+2}f_{n+\frac{3}{2}}+32h^2y_nf_{n+2}f_{n+\frac{3}{2}}+32h^2f_nf_{n+2}y_{n+\frac{5}{4}}\\-110h^2f_ny_{n+2}f_{n+\frac{5}{4}}-12h^2y_nf_{n+2}f_{n+\frac{5}{4}}+30h^3f_nf_{n+\frac{3}{2}}f_{n+\frac{5}{4}}\\-96h^2f_nf_{n+\frac{3}{2}}y_{n+\frac{5}{4}}+140h^2f_nf_{n+\frac{5}{4}}y_{n+\frac{3}{2}}-14h^2y_nf_{n+\frac{3}{2}}f_{n+\frac{5}{4}}\\+32hf_{n+2}y_{n+\frac{3}{2}}y_{n+\frac{5}{4}}-96hy_{n+2}f_{n+\frac{3}{2}}y_{n+\frac{5}{4}}+64hy_{n+2}f_{n+\frac{5}{4}}y_{n+\frac{3}{2}}\\+16hf_ny_{n+2}y_{n+\frac{3}{2}}-32hy_nf_{n+2}y_{n+\frac{3}{2}}+86hy_ny_{n+2}f_{n+\frac{3}{2}}\\-48hf_ny_{n+2}y_{n+\frac{5}{4}}-32hy_nf_{n+2}y_{n+\frac{5}{4}}-37hy_ny_{n+2}f_{n+\frac{5}{4}}\\+32hf_ny_{n+\frac{3}{2}}y_{n+\frac{5}{4}}+64hy_nf_{n+\frac{3}{2}}y_{n+\frac{5}{4}}-52hy_nf_{n+\frac{5}{4}}y_{n+\frac{3}{2}}\\+6h^3f_{n+2}f_{n+\frac{3}{2}}f_{n+\frac{5}{4}}-32h^2f_{n+2}f_{n+\frac{3}{2}}y_{n+\frac{5}{4}}+12h^2f_{n+2}f_{n+\frac{5}{4}}y_{n+\frac{3}{2}}\\+20h^2y_{n+2}f_{n+\frac{3}{2}}f_{n+\frac{5}{4}}\end{array}\right)}{\left(\begin{array}{c}32y_ny_{n+2}-72y_ny_{n+\frac{3}{2}}+40y_ny_{n+\frac{5}{4}}+40y_{n+2}y_{n+\frac{3}{2}}-72y_{n+2}y_{n+\frac{5}{4}}\\+32y_{n+\frac{3}{2}}y_{n+\frac{5}{4}}-32hf_ny_{n+2}+48hf_ny_{n+\frac{3}{2}}+42hy_nf_{n+\frac{3}{2}}\\-16hf_ny_{n+\frac{5}{4}}-39hy_nf_{n+\frac{5}{4}}+6h^2f_{n+\frac{3}{2}}f_{n+\frac{5}{4}}-36h^2f_nf_{n+\frac{3}{2}}\\+30h^2f_nf_{n+\frac{5}{4}}-10hy_{n+2}f_{n+\frac{3}{2}}+27hy_{n+2}f_{n+\frac{5}{4}}-32hf_{n+\frac{3}{2}}y_{n+\frac{5}{4}}\\+12hf_{n+\frac{5}{4}}y_{n+\frac{3}{2}}\end{array}\right)} \qquad 10$$

respectively.

STABILITY PROPERTIES OF THE METHOD

Order of the Method*:* We associate the operator ℓ with the non-linear defined by:

$$\ell\left[y(x):\ h\right]=y_{n+t}-y(x_{n+t})=0 \qquad \textbf{(11)}$$

where $y\left(x_n\right)$ is an arbitrary function continuously differentiable on $\left[a,b\right]$. Following Fatunla (1980), we can write terms in (11) as a Taylor series expansion about the point x to obtain the expansion:

$$\ell\left[y(x):\ h\right]=c_0y(x)+c_1hy'(x)+...+c_ph^py^p(x)+... \qquad \textbf{(12)}$$

where the constant coefficients c_p, $p=0,1,2,...$ are given as:

$$c_p=\frac{1}{p!}\left[\sum_{j=1}^{r}j^p\Phi_j-\frac{1}{(p-1)!}\sum_{j=1}^{r}j^{p-1}\Psi_j-\frac{1}{(p-2)!}\sum_{j=1}^{r}j^{p-2}\Gamma_j\right]$$

(12) has order p if

$$\ell\left[y(x):\ h\right]=0\left(h^{p+1}\right),c_0=c_1=...=c_{p+1}\neq 0$$

therefore c_{p+1} is the error constant and $c_{p+1}h^{p+1}y^{p+1}$ is the local truncation error (LTE).

Zero Stability: A numerical method is said to be zero stable if $\lim_{h\to 0} y_{n+u} = y_n$ where w is the evaluation point.

Consistency: A numerical method is said to be consistent if:

(i) it has order $p \geq 1$

(ii) $\lim_{h\to 0}\left(\frac{1}{h}\left(y_{n+w} - y_n\right)\right) = wy_n^{'}$

Convergent: A numerical method is said to be convergent.

(i) $\lim_{h\to 0}\left(y(x) - y_n(x)\right) \to 0$ where $y(x)$ is the exact solution and $y_n(x)$ is the approximate solution.

(ii) if it is consistent and zero stable.

Stability: A numerical method is said to be A-stable if $\lim_{z\to\infty} R(z) \leq 1$

Results of the Stability Properties

Substituting the test equation $y' = \lambda y$ into (9) and (10) gives the results shown in Table 1.

NUMERICAL EXAMPLES

We consider the following problems to test the efficiency of the developed method. The following notation are used in the tables below; $y_i,\ i = 1,2$, y_n is the exact solution, y is the computed result at each case, $|error|_i, i = 1,2$ is the absolute error given by $(y - y_n)$ at each case and max $|error|_i, i = 1,2$ is the maximum absolute error computed.

Example 1:

$$\begin{pmatrix} y_1^{'}(x) \\ y_2^{'}(x) \end{pmatrix} = \begin{pmatrix} -y_1(x) + 95y_2(x) \\ -y_1(x) - 97y_2(x) \end{pmatrix}, \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The eigenvalues of the Jacobian matrix of the system are $\lambda_1 = -2$ and $\lambda_2 = -96$ with stiffness ratio 48. The exact solution is given as

$$\begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} = \frac{1}{47}\begin{pmatrix} 95\exp(-2x) - 48\exp(-96x) \\ 48\exp(-96x) - \exp(-2x) \end{pmatrix}$$

Source: Okunuga (1997).

Table 1: Local Truncation Error.

CNLM	Order	LTE
9	8	$\frac{161}{5760}h^9(-40(y_n''')^3 - 18(y_n'')^2y_n^5 - 15y_n'y_n^8 + 12y_n'y_n'''y_n^5 + 60y_n''y_n'''y_n^4)$
10	8	$-\frac{13}{384}h^9(-40(y_n''')^3 - 18(y_n'')^2y_n^5 - 15y_n'y_n^8 + 12y_n'y_n'''y_n^5 + 60y_n''y_n'''y_n^4)$

CNLM is the continuous nonlinear method and LTE is the results of the local truncation error.

Table 2: Comparison of Example 1 Results.

h	$\lvert error\rvert_1$	$\lvert error\rvert_2$	OKG (1997)
0.0625	2.8973(−007)	8.9614(−007)	9.1766(−005)
0.03125	2.1707(−008)	1.8758(−008)	9.6596(−007)
0.05	4.4126(−009)	4.2561(−009)	4.1662(−009)

Example 2:

$$\begin{pmatrix} y_1'(x) \\ y_2'(x) \end{pmatrix} = \begin{pmatrix} -1002y_1(x)+1000y_2^2(x) \\ y_1(x)-y_2(x)(1+y_2(x)) \end{pmatrix}, \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The exact solution:

$$\begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} = \begin{pmatrix} \exp(-2x) \\ \exp(-x) \end{pmatrix}$$

Source: Wu and Xia (2001)

Solving this problem at $h = 0.01$, gives the result below:

Table 3: Comparison of Example 2 Result.

x	y_i	$\lvert error\rvert_1$	$\lvert error\rvert_2$	W & X (2001)
1	y_1	2.6563(−010)	1.0506(−009)	2.5606(−007)
	y_2	1.2983(−008)	1.3326(−008)	8.0150(−008)

Example 3:

$$y' = 1 + y^2$$

With $y(0) = 1$, $x \in [0, 0.7]$ and $h = 0.001$ the exact solution is:

$$y = \tan(t + \frac{\pi}{4})$$

Source: Ronald and Thuso (2014)

Table 4: Comparison of Example 3 Result.

x	Max$\lvert error\rvert_1$	Max$\lvert error\rvert_2$	R&T (2014)
0.1	5.5767(−04)	1.9470(−05)	3.2221(−04)
0.2	2.7526(−04)	2.2106(−05)	2.4712(−04)
0.3	8.9453(−05)	1.7065(−06)	1.4958(−04)
0.4	1.4399(−05)	8.3233(−06)	9.3454(−05)
0.5	4.3334(−06)	3.1284(−06)	5.7570(−05)
0.6	7.0045(−06)	2.1172(−07)	1.0152(−05)
0.7	1.6787(−07)	7.6218(−08)	1.0606(−06)
0.8	5.2394(−07)	1.8354(−08)	5.5600(−07)
0.9	8.0897(−07)	3.2536(−08)	2.3530(−07)
1	2.2567(−08)	7.6541(−09)	1.5270(−07)

Example 4:

$$y'(x) = -10xy$$

with $y(0) = 1$ and the exact solution is given as:

$$y(x) = e^{-5x^2}$$

Source: Musa, Suleiman and Senu (2012)

Table 5: Comparison of Example 4 Result.

h	Max$\lvert error\rvert_1$	Max$\lvert error\rvert_2$	Musa & Senu (2012)
10^{-2}	1.9601(−004)	2.0356(−004)	1.2408(−002)
10^{-3}	1.7316(−006)	1.2386(−006)	7.3642(−004)
10^{-4}	2.5672(−007)	1.4258(−007)	7.0552(−005)
10^{-5}	2.1534(−008)	1.4857(−007)	7.0355(−006)
10^{-6}	1.1748(−009)	5.6622(−009)	7.0326(−007)

DISCUSSION OF RESULT

A two-step hybrid method for the solution of stiff and singular first order initial value problems is developed. The two-step is then partitioned into two points $u \, and \, v$ for two cases (I and II).

Table 1, shows the stability results for both case I and case II, it has been observed that the method is of order 8.

In Table 2, for easy comparison, the values of h is taken as it is in Okunuga (1997). h varies from 0.0625 to 0.03125 and 0.05. It is observed that the two cases performed better.

In Table 3, comparison between the error of CLNM and Wu and Xi (2001) shows both the two cases are better with minimal error.

Table 4 shows that x is given as 0:0.1:1 as it is in Ronald and Thuso (2014). CNLM gives better results too.

In Table 5, h is decreases from 10^{-2} to 10^{-6}, in the results obtained, it shows that our method has minimal error compared to Musa, Suleiman, and Senu (2012).

It can be seen from the results in each table and from the stability analysis of the new method competes with the existing method and finally the developed method is zero stable, consistent and converges faster.

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