On Hybrid Block Methods for the Solution of Optimal Control Problems

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ABSTRACT

This paper considers an indirect method for the solution of optimal control problems modelled in ordinary differential equations using hybrid linear multistep block methods via Pontryagin's principle. Interpolation and collocation approach is adopted for the development of the method. The method developed is consistent, convergent and was found to compete favorably with the existing classical Runge Kutta method.

(Keywords: optimal control, interpolation, collocation, convergent)

INTRODUCTION

Many physical problems in our environment today give rise to differential equations. Traditionally, solutions to these differential equations can be obtained using analytical methods. However, solutions to certain differential equations are very difficult (Areo and Adeniyi, 2013).

Optimal control is a subject where it is desired to determine the inputs to a dynamical system that optimize (minimize or maximize) a specific performance index while satisfying certain constraints on the motion of the system (Rose, 2015). Optimal control models have been used in biological problems, notably in population interactions and diseases. The models range from discrete to systems of ordinary differential equations and partial differential equations, with variable applications such as HIV, vector-borne diseases and pest control (Blayneh and Ngnepieba, 2012).

This paper, considers Optimal Control Problem (OCP) of the form:

$$\max_{u(\cdot)} \; J\left[x(\cdot),u(\cdot)\right]=\int_{t_0}^{t_f} f(x,x(t),u(t))dt \qquad \textbf{(1)}$$

subject to dynamic system:

$$\begin{aligned} &\dot{x}'=g(t,\ x(t),\ u(t)),\\ &x(t_0)=x_0,\ \ t_f \ \text{ is free} \end{aligned} \qquad \textbf{(2)}$$

where $x(t)$ and $u(t)$ are real valued functions, f and g are continuously differentiable functions. A dynamic system is mathematically characterized by a set of Ordinary Differential Equations (ODE). Specifically, the dynamics are described for $t_0 \le t \le t_f$, by a system of (ODEs):

$$\dot{y}=\left[y_i\right]=\left[\,f_i(y_1(t),...,y_n(t),t)\right] \qquad \textbf{(3)}$$

where $i=1,2,...,n.$ The differential equation of (2) describes the dynamics of the system while the performance index is a measure of the quality of the trajectory. When it is desired to minimize the performance index, a lower value of J is better, conversely, when it is desired to maximize the performance index, a higher value of J is better. Rodrigues, Monteiro and Torres (2014) applied Forward Backward Sweep method to solve optimal control problem using classical fourth order Runge Kutta via Pontryagin's principle. In this paper we improved on Rodrigues, Monteiro and Torres (2014) approach by using indirect method to develop hybrid block method that solve optimal control problems.

Many scholars have developed different methods for solving optimal control problems of the form (1), among them are: Adesanya, Odekunle, and Adeyeye (2012), Adesanya, Alkali, and Sunday (2014), Akinfenwa and Okunuga (2014). The authors individually implemented their methods such that it gives better solution. Recently, Ajileye, Amoo, and Ogwumu (2018) and Abdelrahim and Omar (2016) developed hybrid block methods for the solution of initial value problems (IVPs) in Ordinary Differential Equations. Their methods was found to be convergent and absolutely stable with large region of absolute stability.

Optimal Control System

Lenhart and Workman (2007) considered Optimal Control (OC) problem of the form (1,2). The solution must satisfy the following conditions:

$$x'(t) = g(t, x(t)),\ x(t_0) = x_0,$$

$$\lambda'(t) = -\frac{\partial H}{\partial x} = -(f_x(t,x,u) + \lambda(t) g_x(t,x,u)),\ \lambda(t_1) = 0,$$

$$\frac{\partial H}{\partial u} = f_u(t,x,u) + \lambda(t) g_u(t,x,u) \quad \text{at } u^*.$$

The OC can be manipulated to find a representation of u^* in terms of t, x and λ, where u^* is the optimal point. If this representation is substituted back into the ODEs for x and λ, then the first two equations form a two-point boundary value problem.

Necessary Conditions of Optimality in OC

(i) State Equation $\quad x = \dfrac{\partial H}{\partial \lambda} = f(t,x,u)$

(ii) Constrain Equation

$$\lambda = -\left(\frac{\partial H}{\partial x}\right) = g(t,x,u)$$

(iii) Optimal Control Equation

$$\left(\frac{\partial H}{\partial u} = 0\right) \Rightarrow u = \psi(x, \lambda)$$

MATHEMATICAL BACKGROUND

Development of the Method

We consider a polynomial approximate solution of the form:

$$y(x) = \sum_{n=0}^{k} a_n x^n \qquad \textbf{(4)}$$

with first derivative given as:

$$y'(x) = \sum_{n=1}^{k} n a_n x^{n-1} \qquad \textbf{(5)}$$

where $k = s + r - 1$. $k \in [0, r]$,

where r and s are the numbers of interpolation and collocation points, respectively. Interpolating (4) at x_{n+k}; and collocating (5) at x_{n+k} gives a system of nonlinear equation of the form;

$$U = AX \qquad \textbf{(6)}$$

where

$$A = [a_0, a_1, a_3, \ldots, a_n] \qquad \textbf{(7)}$$

$$U = [y_n, y_{n+1}, \ldots, y_{n+r}, f_n, f_{n+1}, \ldots, f_{n+s}]^T \qquad \textbf{(8)}$$

$$X = \begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & \cdots & x_n^k \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & \cdots & x_{n+1}^k \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n+r} & x_{n+r}^2 & x_{n+r}^3 & \cdots & x_{n+r}^k \\ 0 & 1 & 2x_n & 3x_n^2 & \cdots & kx_n^{k-1} \\ 0 & 1 & 2x_{n+1} & 3x_{n+1}^2 & \cdots & kx_{n+1}^{k-1} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & 2x_{n+s} & 3x_{n+s}^2 & \cdots & kx_{n+s}^{k-1} \end{bmatrix} \qquad \textbf{(9)}$$

Imposing the following conditions on (4) and (5) respectively:

$$y(x_{n+j}) = y_{n+j},\ j = 0,1,2,3,\ldots,r$$

$$y'(x_{n+j}) = f_{n+j},\ j = 0,1,2,3,\ldots,s$$

and solving (6) using Crammer's rule and then substituting into (4) gives the continuous nonlinear method. Evaluating at the selected grid point using Self Starting techniques gives a discrete solution (Olanegan, Ogunware, Omole, Oyinloye, and Enoch, 2015).

Specification of the Method

To generate the hybrid collocation points, we consider arbitrary collocation points. Evaluating (4) at point x_n and (5) at points $[x_n, x_{n+\frac{1}{2}}, x_{n+1}]$ to give:

$$\begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 \\ 0 & 1 & 2x_n & 3x_n^2 \\ 0 & 1 & 2x_{n+\frac{1}{2}} & 3x_{n+\frac{1}{2}}^2 \\ 0 & 1 & 2x_{n+1} & 3x_{n+1}^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} y_n \\ f_n \\ f_{n+\frac{1}{2}} \\ f_{n+1} \end{bmatrix}$$

The continuous scheme is given as:

$$y_{n+t} = \alpha_0 y_n(t) + \beta_0 f_n(t) + \beta_{\frac{1}{2}} f_{n+\frac{1}{2}}(t) + \beta_1 f_{n+1}(t) \quad \textbf{(10)}$$

where

$$\alpha_0 = 1, \quad \beta_0 = \tfrac{1}{6} t\left(-9t + 4t^2 + 6\right),$$

$$\beta_{\frac{1}{2}} = -\tfrac{2}{3} t^2 \left(2t - 3\right), \quad \beta_1 = \tfrac{1}{6} t^2 \left(4t - 3\right)$$

to obtain the discrete schemes:

When $t = \frac{1}{2}$

$$y_{n+\frac{1}{2}} = y_n + \frac{5}{24} hf_n + \frac{1}{3} hf_{n+\frac{1}{2}} - \frac{1}{24} hf_{n+1} \quad \textbf{(11)}$$

When $t = 1$

$$y_{n+1} = y_n + \frac{1}{6} hf_n + \frac{2}{3} hf_{n+\frac{1}{2}} + \frac{1}{6} hf_{n+1} \quad \textbf{(12)}$$

Writing (11) and (12) in block, to get:

$$A^{(1)} Y_{m+1} = A^{(0)} Y_m + hB^{(0)} F_m + hB^{(1)} F_{m+1} \quad \textbf{(13)}$$

Where,

$$Y_{m+1} = \left[y_{n+\frac{1}{2}} \;\; y_{n+1} \right]^T, Y_m = \left[y_{n-1} \;\; y_n \right]^T,$$

$$F_m = \left[f_{n-1} \;\; f_n \right]^T, \; F_{m+1} = \left[f_{n+\frac{1}{2}} \;\; f_{n+1} \right]^T$$

$$A^{(1)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; A^{(0)} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix},$$

$$B^{(0)} = \begin{bmatrix} 0 & \frac{5}{24} \\ 0 & \frac{1}{6} \end{bmatrix}, \; B^{(1)} = \begin{bmatrix} \frac{1}{3} & -\frac{1}{24} \\ \frac{2}{3} & \frac{1}{6} \end{bmatrix}$$

Stability Properties

We consider the basic properties of the developed method which include order, consistency, zero-stability, convergent and the region of absolute stability.

Order of the Method

Let the linear operator $L\{y(x):h\}$ associated with the block (13) be defined as:

$$L\{y(x):h\} = A^{(0)} Y_m - Ey_n - hdf(y_n) - hbF(Y_m) \quad \textbf{(14)}$$

Expand using Taylor series and compairing the coefficients of h we obtain:

$$L\{y(x):h\} = c_0 y(x) + c_1 hy'(x) + c_2 h^2 y''(x) + \ldots c_p h^p y^p(x) + \cdots \quad \textbf{(15)}$$

The linear operator L and the associated continuous linear multistep method (10) are said to be of order p if,

$$c_0 = c_1 = c_2 = \ldots = c_p = 0 \text{ and } c_{p+1} \neq 0.$$

c_{p+1} is called the error constant (Adesanya, Alkali and Sunday, 2014) and the Local Truncation Error is given by:

$$t_{n+k} = c_{p+1} h^{(p+1)} y^{(p+1)}(x_n) + 0(h^{r+2}) \qquad \textbf{(16)}$$

Consistency

A block method is consistent if it has order $p \geq 1$. Clearly $p \geq 1$ in this case.

Zero-Stability

A block in (13) is said to be Zero-stable, if the roots of the first characteristics polynomial $\rho(z)$ is defined by $\rho(z) = \det(z\tau^{(1)} - \tau^{(0)})$ satisfies $|z| \leq 1$ and the roots $|z| \leq 1$ have multiplicity not greater than the order of the differential equation (Adesanya, Alkali, and Sunday, 2014).

Convergence

A method is said to be convergent if it is consistent and zero-stable. The developed method is of order 5 and the error constant is given as:

$$\eta(z) = \left[\frac{1}{384} h^4, \ -\frac{1}{2880} h^5 \right]^T;$$

it is zero stable, consistent and convergent.

The region of absolute stability of the developed method is shown in Figure 1.

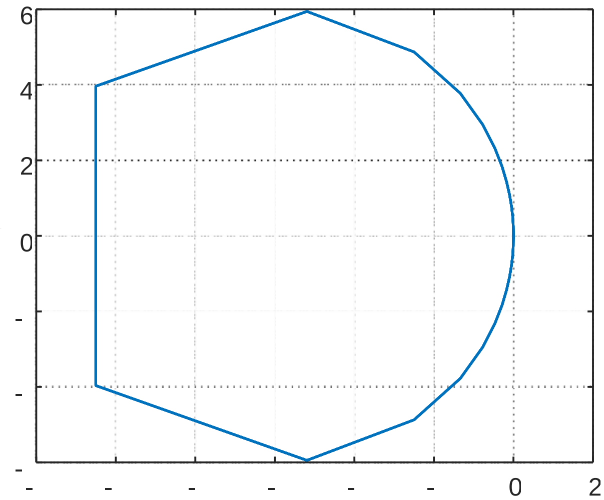

Figure 1: Region of Absolute Stability.

NUMERICAL EXAMPLES

Example 1 (Lenhart & Workman, 2007)

$$\max_u \int_0^1 x(t) + u(t)\,dt$$

$$\text{subject to } \begin{cases} x'(t) = 0.5x(t) + u(t), \\ x(0) = 1, \end{cases}$$

with the optimal solution:

$$x^*(t) = \frac{2e^{3t} + e^3}{e^{\frac{3t}{2}}(2 + e^3)}, \quad u^*(t) = \frac{2(e^{3t} - e^3)}{e^{\frac{3t}{2}}(2 + e^3)}$$

Example 2 (Rose, 2015)

$$\min_u \frac{1}{2} \int_0^1 x(t)^2 + u(t)^2\,dt$$

$$\text{subject to } \begin{cases} x'(t) = -x(t) + u(t) \\ x(0) = 1 \end{cases}$$

with the optimal solution:

$$x(t)=\frac{\sqrt{2}\cosh\left(\sqrt{2}(t-1)\right)-\sinh\left(\sqrt{2}(t-1)\right)}{\sqrt{2}\cosh\left(\sqrt{2}\right)+\sinh\left(\sqrt{2}\right)},$$

$$\lambda(t)=-\frac{\sinh\left(\sqrt{2}(t-1)\right)}{\sqrt{2}\cosh\left(\sqrt{2}\right)+\sinh\left(\sqrt{2}\right)}$$

<u>Example 3</u> (Rose, 2015)

$$\min \int_0^1 x(t)+u(t)dt$$

$$\text{subject to } \begin{cases} x'(t)=1-u(t), \\ x(0)=1, \end{cases}$$

NOTATIONS

x	Point of Evaluation
HBM	Hybrid Block Method
RKM	Classical Runge Kutta Method
Err	Absolute Error
FBS	Forward Backward Sweep

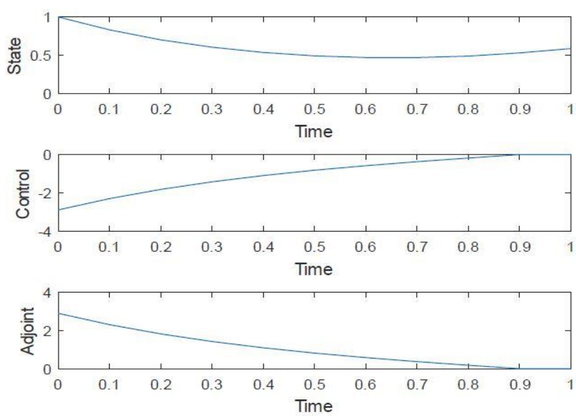

Figure 2: Graphs Produced by the FBS for RKM for example 1 at $t=1$, $x=5.8692\text{e-}01$ $u=0$ $\lambda=0.0000$.

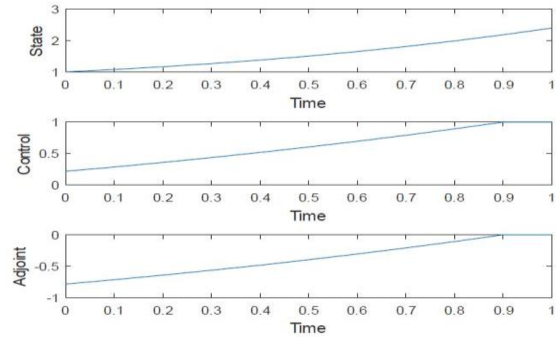

Figure 3: Graphs Produced by the FBS for HBM for example 1 at $t=1$, $x=1.6487\text{e+}00$ $u=0.0000$ $\lambda=0.0000$.

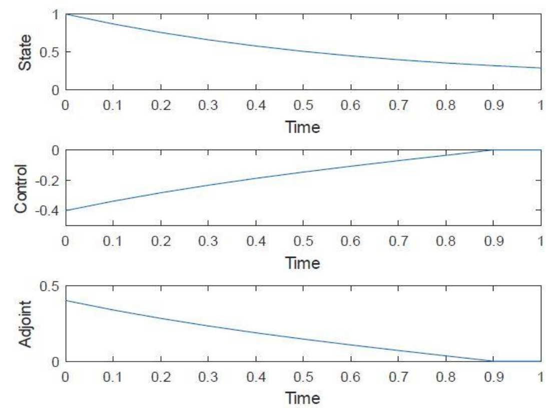

Figure 4: Graphs Produced by the FBS for RKM for example 2 at $t=1$, $x=2.8744\text{e-}01$ $u=0.0000$ $\lambda=0.0000$.

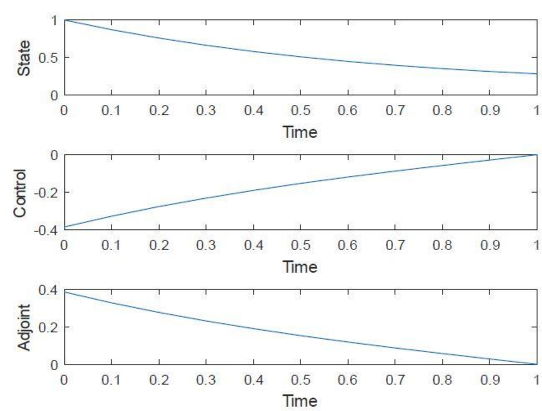

Figure 5: Graphs Produced by the FBS for HBM for example 2 at $t=1$, $x=2.8177\text{e-}01$ $u=0.0000$ $\lambda=0.0000$.

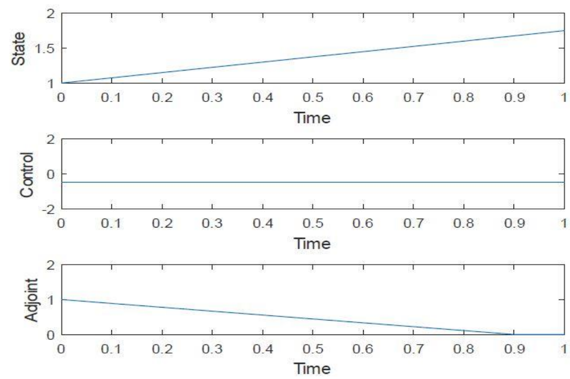

Figure 6: Graphs Produced by the FBS for RKM for example 3 at

$t=1,\ x=$1.7510e+00 $u=$4.9951e-01

$\lambda=0.0000$.

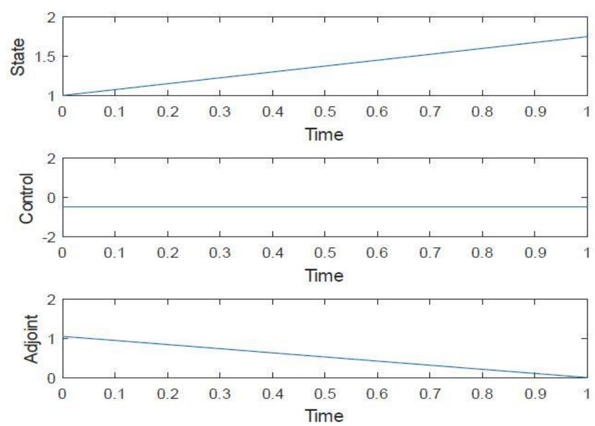

Figure 7: Graphs Produced by the FBS for HBM for example 3 at $t=1,\ x=$1.7510e+00 $u=$4.9951e-01 $\lambda=0.0000$.

DISCUSSION OF RESULTS

For easy comparison, we are using the same values of x, u, t *and* λ as in the work of Lenhart and Workman (2007), and Rose (2015). We incorporate one off-grid point into one step method, and the resulted method is consistent and convergent and it was found to be of order $p = 5$. The region of absolute stability displayed in Figure 1 shows that the method has a large region of absolute stability.

Three examples were used to test the efficiency of the developed method. The results of example 1 is presented in Figures 2 and 3. It shows that the developed method compete effectively with that of Lenhart and Workman (2007). Example 2 and 3 were compared with that of Rose (2015). The results were presented in Figures 4, 5, 6, and 7, respectively. The developed method competes favorably.

It is clearly observed that, HBM developed performed better than the existing RKM in Lenhart and Workman (2007) and Rose (2015). Hence the developed method is efficient and computationally reliable and economical to implement.

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