Radial Basis Function - Finite Difference (RBF-FD) Approximations for Two-Dimensional Heat Equations

M.O. Durojaye and J.K. Odeyemi

Department of Mathematics, University of Abuja, Abuja, Nigeria.

E-mail: mayojaye@yahoo.com*

ABSTRACT

Radial Basis Function-Finite Difference (RBF-FD) approximations for two-dimensional heat equation were formulated using four infinitely smooth positive definite radial basis functions (RBFs), namely, Gaussian (GA), Multiquadrics (MQ), Inverse Multiquadrics (IMQ) and Inverse Quadratic (IQ).

The RBFs were used for discretizating the space variables while Runge-Kutta method was used as a time-stepping method to integrate the resulting systems of differential equations. The accuracy of the RBF-FD discretization can be increased by considering not only the data sites but also the derivative values on the nodes present in the supporting region. The RBF-FD approach makes it easy to test how different node distributions influence the extent to which the physics was captured. The numerical results and its comparison with other numerical methods are presented.

(Keywords: Radial Basis Function, Gaussian (GA), Multiquadrics (MQ), Inverse Multiquadrics (IMQ) and Inverse Quadratic (IQ))

INTRODUCTION

Finite difference methods are numerical techniques for finding solutions to PDEs that approximate the solution on a mesh of points that are equally spaced across the domain. In situations where the preferred points are not on a mesh, or when the domain does not give itself to simple meshes, it is desirable to have mesh-free methods. Meshless methods are one such class of methods in which the solution is approximated on a set of nodes with no specified connectivity. Radial Basis Function Finite Difference methods (RBF-FD) are such mesh-free methods.

Recently, a RBF-FD method was proposed by Flyer and Fornberg [16]. The RBF-FD method generates a local RBF interpolant for expressing the function derivatives at a node as a linear combination of the function values on the nodes present in the neighborhood of the considered node. Also, this RBF interpolants are used to generate the weights of a Finite Difference (FD) formula [5].

The RBF-FD concept [3, 12, 14, 15] is to require such approximations to be exact for radial basis functions (RBFs) rather than for multivariate polynomials. For infinitely smoothed RBFs, this procedure can never give rise to singularities, no matter how the nodes are distributed [7, 9, 10]. The outcomes tend to become mostly accurate when using nearly flat RBFs with a small shape parameter ε) [1, 6, 8, 13], but the resulting systems will then again become ill conditioned. However, in contrast to the multivariate polynomial case, the ill-conditioning that arises in the RBF case is not of a fundamental nature, and it can be avoided by using appropriate numerical algorithms [5].

The latest RBF-FD studies described numerical solutions of elliptic and of convective-diffusive PDEs. The approach was soon afterwards shown to be well suited for computational fluid mechanics [2, 11, 13], more recently also in purely convective situations [4, 10].

This paper studies the accurate computation of the weights (coefficients) in RBF-FD formulas for two-dimensional heat equations.

Radial Basis Functions

Radial basis functions approximate a function f(x) sampled at some set of N distinct node locations by translates of a single radially symmetric function ϕ(r). Given the data values $\{f_k\}_{k=1}^{N}$ at the node locations $\{\mathbf{x}_k\}_{k=1}^{N}$, the Radial basis function interpolant s(x) to the data is defined by:

$$s(\mathbf{x}) = \sum_{k=1}^{N} c_k \phi(\|\mathbf{x} - \mathbf{x}_k\|) \qquad \textbf{(1)}$$

where the expansion coefficients, $\{c_k\}_{k=1}^{N}$, are established by applying the collocation conditions such that the residual is zero at the data locations. This is equivalent to solving the symmetric linear system of equations:

$$\underbrace{\begin{bmatrix} \phi(\|\mathbf{x}_1 - \mathbf{x}_1\|) & \phi(\|\mathbf{x}_1 - \mathbf{x}_2\|) & \dots & \phi(\|\mathbf{x}_1 - \mathbf{x}_N\|) \\ \phi(\|\mathbf{x}_2 - \mathbf{x}_1\|) & \phi(\|\mathbf{x}_2 - \mathbf{x}_2\|) & \dots & \phi(\|\mathbf{x}_2 - \mathbf{x}_N\|) \\ \vdots & \vdots & \ddots & \vdots \\ \phi(\|\mathbf{x}_N - \mathbf{x}_1\|) & \phi(\|\mathbf{x}_N - \mathbf{x}_2\|) & \dots & \phi(\|\mathbf{x}_N - \mathbf{x}_N\|) \end{bmatrix}}_{A} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_N \end{bmatrix} \qquad \textbf{(2)}$$

where A is the interpolation matrix. Studies have shown that for radial basis functions such as the Gaussian, inverse multiquadric, and inverse quadratic, the matrix in (2) is positive definite regardless of the distinct node locations and the dimension. Likewise, if the shape parameter, ϵ, is kept fixed throughout the domain best results are realized with roughly evenly distributed nodes [12].

Numerous examples of frequently used RBFs are shown in Table 1 A clear difference is made here between three different categories: the infinitely smooth, the piecewise smooth and the compactly supported RBFs. Radial basis functions of the first type are $C^{\infty}(0,\infty)$ and can provide spectral accuracy, while the piecewise smooth RBFs give algebraic convergence for interpolation.

RBF-FD Approximations

The RBF-FD approximation is analogous in concept to classical finite-differences (FD), in that derivatives of a function $u(x)$ are approximated by weighted combinations of n function values in a trivial neighborhood around a single center node, x_c. That is:

$$\mathcal{L}u(x)|_{x=x_c} \approx \sum_{j=1}^{n} w_j u(x_j) \qquad \textbf{(3)}$$

Table 1: Examples of Commonly used Radial Basis Functions.

Name	$\phi(r)$	Notes
	Infinitely smooth	
Gaussian (GA)	$e^{-(\varepsilon r)^2}$	$\varepsilon \in \mathbb{R}$
Multiquadrics (MQ)	$\sqrt{1+(\varepsilon r)^2}$	$\varepsilon \in \mathbb{R}$
Inverse Multiquadrics (IMQ)	$\frac{1}{\sqrt{1+(\varepsilon r)^2}}$	$\varepsilon \in \mathbb{R}$
Inverse Quadratic (IQ)	$\frac{1}{1+(\varepsilon r)^2}$	$\varepsilon \in \mathbb{R}$
Generalized Multiquadric	$(1+(\varepsilon r)^2)^{\beta}$	$\varepsilon \in \mathbb{R}, \beta \in \mathbb{R}\backslash\mathbb{N}_0$
Laguerre–Gaussian	$(2-(\varepsilon r)^2)e^{-(\varepsilon r)^2}$	$\varepsilon \in \mathbb{R}^2$, oscillatory
Poisson	$\sqrt{\frac{\pi}{2}}\frac{\sin(\varepsilon r)}{\varepsilon r}$	$\varepsilon \in \mathbb{R}^3$, oscillatory
	Piecewise smooth	
Radial power	r^{β}	$0 < \beta \notin 2\mathbb{N}$
Thin plate spline	$r^{2\beta}\log r$	$\beta \in \mathbb{N}$
	Compactly supported	
Wendland	$(1-r)_+^6(35r^2+18r+3)$	Ex., C^4 in $\mathbb{R}^3$
Gneiting	$(1-r)_+^5(1+5r-27r^2)$	Ex., C^2 in $\mathbb{R}^2$

where $\mathcal{L}u$ represents a differential operator on $u(x)$ (e.g., $\mathcal{L} = \frac{\partial}{\partial x}$ or $\mathcal{L} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$).

Here the n nodes are known as a stencil with size n. The w_j are stencil weights. In practice stencils include the center, x_c, plus the $n-1$ nearest neighboring nodes. The definition of "nearest" depends on the Euclidean distance ($\|x - x_c\|_2$).

In a situation whereby the 2-D nodes are not situated on a regular grid, the weights for approximating an operator $\mathcal{L}$ (such as $\mathcal{L} = \frac{\partial}{\partial x}$ or $\mathcal{L} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$) can no longer be obtained by means of 1-D polynomial procedures. This can be overcome by enhancing bivariate polynomials with radial basis functions $\phi(\|\boldsymbol{x} - \boldsymbol{x_k}\|_2)$, with one such centered at each stencil point $\boldsymbol{x_k} = (x_k, y_k)$, and then introducing matching constraints to the associated RBF expansion coefficients.

The weights for scattered node RBF-FD stencils can then be obtained by solving a linear system of the form indicated in (4) in the case of using up through linear polynomials in x and y:

$$\left[\begin{array}{ccc|ccc} & \vdots & & 1 & x_1 & y_1 \\ & A & & \vdots & \vdots & \vdots \\ & \vdots & & 1 & x_n & y_n \\ \hline 1 & \dots & 1 & & & \\ x_1 & \dots & x_n & & 0 & \\ y_1 & \dots & y_n & & & \end{array}\right] \left[\begin{array}{c} \vdots \\ w_1 \\ \vdots \\ w_n \\ \hline w_{n+1} \\ w_{n+2} \\ w_{n+3} \end{array}\right] = \left[\begin{array}{c} \vdots \\ \mathcal{L}\phi(\|\boldsymbol{x} - \boldsymbol{x_1}\|_2)|_{x=x_c} \\ \vdots \\ \mathcal{L}\phi(\|\boldsymbol{x} - \boldsymbol{x_n}\|_2)|_{x=x_c} \\ \hline \mathcal{L}1|_{x=x_c} \\ \mathcal{L}x|_{x=x_c} \\ \mathcal{L}y|_{x=x_c} \end{array}\right] \quad \textbf{(4)}$$

The entries in the matrix A are $A_{i,j} = \phi(\|\boldsymbol{x} - \boldsymbol{x_1}\|_2)$. All the terms in the RHS should be evaluated at the stencil's 'center point' $\boldsymbol{x_c}$. In the solution vector, $w_1, \dots, w_n$ provides the weights to be used at nodes $\boldsymbol{x_k}, k = 1,2, \dots, n$, while the remaining w-entries should be ignored.

In (3), A-matrix is the same as the matrix that arises in RBF interpolation of scattered data $\{\boldsymbol{x_k}, f_k\}, k = 1,2, \dots, n$, for finding the interpolant of the form:

$$r(\boldsymbol{x}) = \sum_{k=1}^{n} \lambda_k \, \phi(\|\boldsymbol{x} - \boldsymbol{x_k}\|_2) + q_m(x) \quad \textbf{(5)}$$

with constraint:

$$\sum \lambda_k \, q_m(x) = 0 \quad \textbf{(6)}$$

where the coefficients λ_k can be found as the solution to the linear system:

$$A\,\lambda = f \quad \textbf{(7)}$$

Here, RBFs are used to calculate derivative approximations in local mode, where the N nodes across the full domain need its own stencil. In the local mode, the number of nodes/weights n within each stencil is less than the total number of nodes N in the full domain.

Differentiation Matrices

For time-dependent PDEs, the stencil weights remain constant for all time-steps when the nodes are stationary because RBF-FD weights are only a function of node locations.

Given the set of nodes in the domain, $\{\mathbf{x}_k\}_{k=1}^{N}$, the w-th row of the differentiation matrix (DM) represents the discrete PDE operator for the stencil centered at node $\mathbf{x}_c$ with stencil nodes $\{\mathbf{x}_j\}_{j=1}^{n}$:

$$\mathcal{L}u(x) \approx D_{\mathcal{L}} u \quad \textbf{(8)}$$

$$D_{\mathcal{L}}^{(w,k)} = \begin{cases} \mathrm{w}_j & \mathbf{x}_k = \mathbf{x}_j \\ 0 & \mathbf{x}_k \neq \mathbf{x}_j \end{cases} \quad \textbf{(9)}$$

where the condition $\mathbf{x}_k = \mathbf{x}_j$ maps the local stencil index for each node, j, to a global index, k, and (w, k) is the global (row, column) index of $D_{\mathcal{L}}$. Vector $u = \{u(\mathbf{x}_k)\}_{k=1}^{N}$. Equation (4) can be rewritten as:

$$\mathcal{L}u(x)|_{x=x_c} \approx D_{\mathcal{L}}^{(w)} u$$

DMs are utilized in both explicit and implicit modes [***]. The explicit mode entails calculating the matrix-vector multiplied to get derivative values, u', from explicitly known vector of solution values u:

$$u' = D_{\mathcal{L}} u \tag{10}$$

whereas implicit solves for unknown u:

$$D_{\mathcal{L}} u = f \tag{11}$$

RBF-FD weights assemble the rows of the differentiation matrix, D_x. The sparsity of rows reflects the subset of $\{\mathbf{x}_k\}_{k=1}^{N}$ included in corresponding stencils of size n. The mapping between global indices (k) and local indices (j) is used for each sum.

On the right-hand side, discrete derivative values $\frac{du}{dx}$ are approximated at all stencil centers. Therefore, the calculation of the differentiation weights is performed once in a single preprocessing step of $O(n^3 N)$ floating point operations (FLOPs) [**].

Consider for example, the 2-D Laplacian operator,

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}: \tag{12}$$

$$\nabla^2 u \approx D_{\nabla^2} u$$

which can be expanded as:

$$\nabla^2 u \approx \left(D_{\mathrm{x}^2} + D_{\mathrm{y}^2}\right) u \approx D_{\mathrm{x}} D_{\mathrm{x}} u + D_{\mathrm{y}} D_{\mathrm{y}} u = D_{\mathrm{x}^2} u + D_{\mathrm{y}^2} u \tag{13}$$

where either a single DM is composed by adding two lower order DMs, or the lower order DMs are directly multiplied against the vector u.

Method of Lines (MOL) Formulation and Time Stepping Considerations for the PDE Test Problem

The full discretized equation of time-dependent PDE is given by:

$$\frac{\mathrm{d}\underline{\mathrm{u}}}{\mathrm{dt}} = D_{\mathrm{x}} D_{\mathrm{x}} \underline{\mathrm{u}} + D_{\mathrm{y}} D_{\mathrm{y}} \underline{\mathrm{u}} \tag{14}$$

evaluated at the nodes $\{\mathbf{x}_k\}_{k=1}^{N}$. The system is advanced in time using the classical Runge-Kutta 4th-order method (RK4).

The standard PDE test problem that we will consider describes two-dimensional temperature diffusion over a plate. Consider a two-dimensional parabolic PDE:

$$10^{-4}\left(\frac{\partial^2 u(x,y,t)}{\partial x^2} + \frac{\partial^2 u(x,y,t)}{\partial y^2}\right) = \frac{\partial u(x,y,t)}{\partial t} \tag{15}$$

for $0 \le x \le 4, 0 \le y \le 4$ and $0 \le t \le 8000$ with the initial conditions and boundary conditions:

$$\begin{aligned} &u(x,y,0) = 0 \\ &u(0,y,t) = e^y - \cos y, \\ &u(x,0,t) = \cos x - e^x, \\ &u(4,y,t) = e^y \cos 4 - e^4 \cos y \\ &u(x,4,t) = e^4 \cos x - e^x \cos 4 \end{aligned} \tag{16}$$

For solving this equation, we have $u(x,y,t)$, which describes the temperature distribution over a square plate having each side 4 units long (shown in Figure 1).

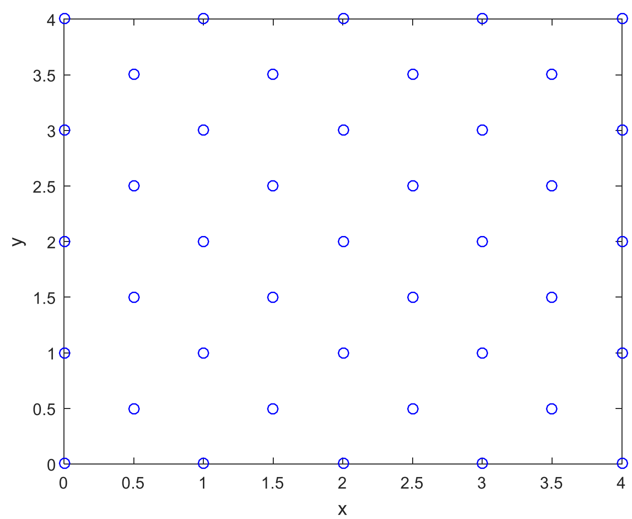

Figure 1: The Temperature Distribution over a Square Plate Having Each Side 4 Units Long.

Table 1: Computation Table.

Computational Methods					Error	
X	**y**	**RBF-FD**	**FDM**	**Exact Solution**	**RBF-FD**	**FDM**
0	0	0	0	0	0.00E+00	0.00E+00
0	1	2.177979523	2.177979523	2.177979523	0.00E+00	0.00E+00
0	2	7.805202935	7.805202935	7.805202935	0.00E+00	0.00E+00
0	3	21.07552942	21.07552942	21.07552942	0.00E+00	0.00E+00
0	4	55.25179365	55.25179365	55.25179365	0.00E+00	0.00E+00
0.5	0.5	1.42E-18	-0.00520602	0	1.42E-18	5.21E-03
0.5	1.5	3.723337739	3.55269058	3.816426247	9.31E-02	2.64E-01
0.5	2.5	11.72480235	11.42495837	12.01200678	2.87E-01	5.87E-01
0.5	3.5	30.17725346	30.34838635	30.60549923	4.28E-01	2.57E-01
1	0	-2.177979523	-2.17797952	-2.177979523	0.00E+00	0.00E+00
1	1	2.58E-16	-0.01765813	0	2.58E-16	1.77E-02
1	2	4.915773147	4.447615015	5.123528432	2.08E-01	6.76E-01
1	3	13.1143209	12.67896622	13.54334053	4.29E-01	8.64E-01
1	4	31.27629394	31.27629394	31.27629394	0.00E+00	0.00E+00
1.5	0.5	-3.723337739	-3.57773119	-3.816426247	9.31E-02	2.39E-01
1.5	1.5	2.61E-16	-0.03337493	0	2.61E-16	3.34E-02
1.5	2.5	4.227789894	3.611470068	4.452232118	2.24E-01	8.41E-01
1.5	3.5	6.345768427	5.995804107	6.539402104	1.94E-01	5.44E-01
2	0	-7.805202935	-7.80520294	-7.805202935	0.00E+00	0.00E+00
2	1	-4.915773147	-4.50261177	-5.123528432	2.08E-01	6.21E-01
2	2	-7.67E-18	-0.05252514	0	7.67E-18	5.25E-02
2	3	-1.105050081	-1.66875601	-1.043422556	6.16E-02	6.25E-01
2	4	-17.89103803	-17.891038	-17.89103803	0.00E+00	0.00E+00
2.5	0.5	-11.72480235	-11.4520324	-12.01200678	2.87E-01	5.60E-01
2.5	1.5	-4.227789894	-3.69818705	-4.452232118	2.24E-01	7.54E-01
2.5	2.5	-4.24E-17	-0.06806797	0	4.24E-17	6.81E-02
2.5	3.5	-14.98112309	-15.4105001	-15.12185498	1.41E-01	2.89E-01
3	0	-21.07552942	-21.0755294	-21.07552942	0.00E+00	0.00E+00
3	1	-13.1143209	-12.7252976	-13.54334053	4.29E-01	8.18E-01
3	2	1.105050081	1.562554065	1.043422556	6.16E-02	5.19E-01
3	3	8.34E-17	-0.05885971	0	8.34E-17	5.89E-02
3	4	-40.92297578	-40.9229758	-40.92297578	0.00E+00	0.00E+00
3.5	0.5	-30.17725346	-30.360523	-30.60549923	4.28E-01	2.45E-01
3.5	1.5	-6.345768427	-6.04062642	-6.539402104	1.94E-01	4.99E-01
3.5	2.5	14.98112309	15.33345151	15.12185498	1.41E-01	2.12E-01
3.5	3.5	3.32E-17	-0.02239967	0	3.32E-17	2.24E-02
4	0	-55.25179365	-55.2517937	-55.25179365	0.00E+00	0.00E+00
4	1	-31.27629394	-31.2762939	-31.27629394	0.00E+00	0.00E+00
4	2	17.89103803	17.89103803	17.89103803	0.00E+00	0.00E+00
4	3	40.92297578	40.92297578	40.92297578	0.00E+00	0.00E+00
4	4	0	0	0	0.00E+00	0.00E+00

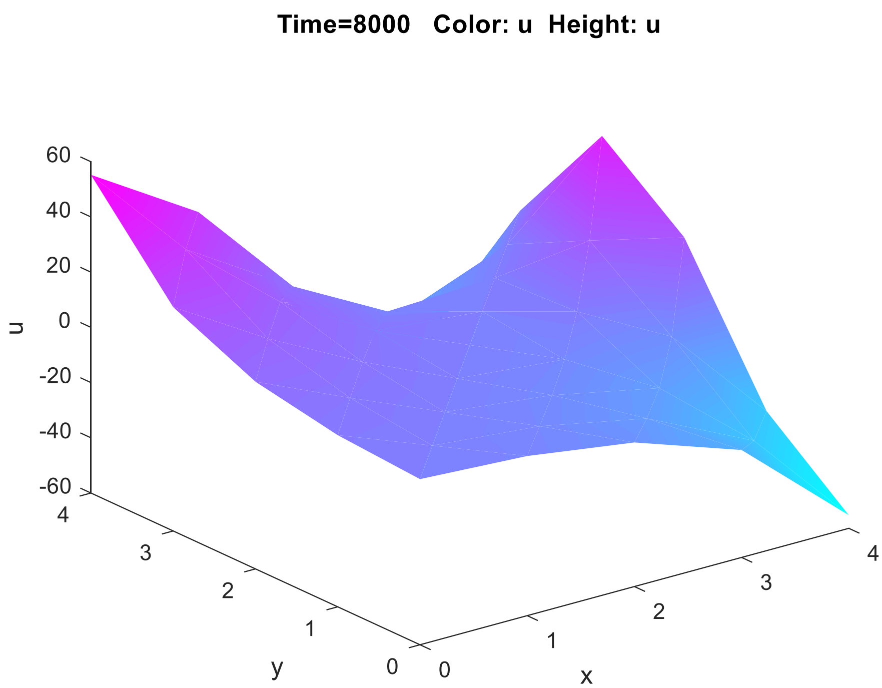

Figure 2: The Solution of a Two-Dimensional Parabolic PDE Obtained using RBF-Finite Difference Method .

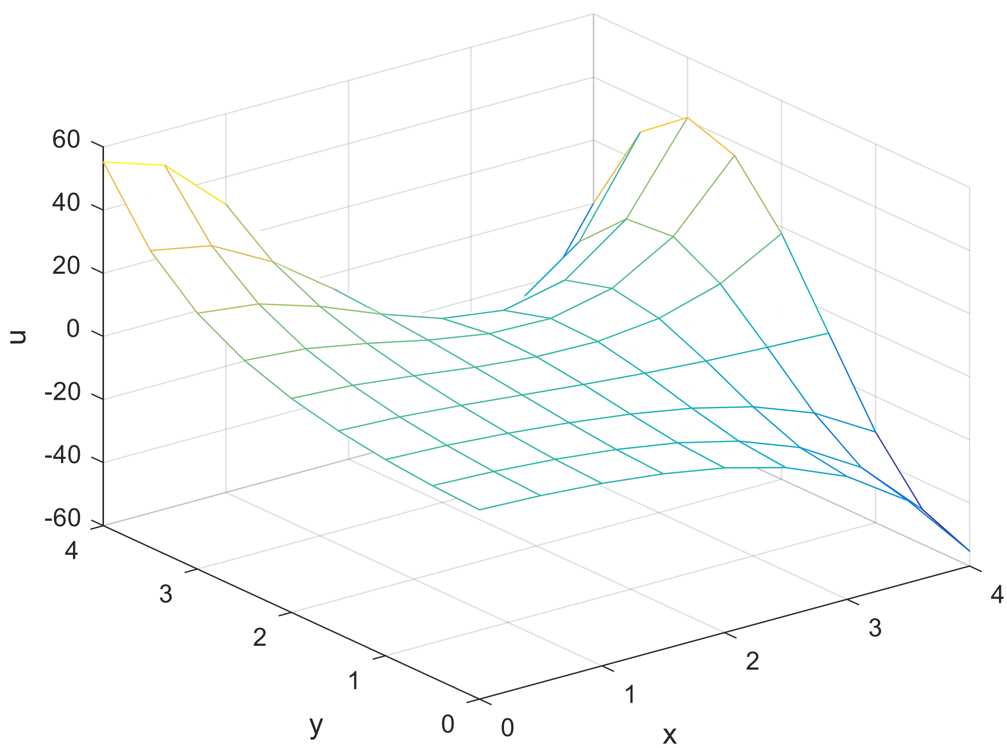

Figure 3: The Solution for a Two-Dimensional Parabolic PDE Obtained using Finite Difference Method.

CONCLUSION

This paper shows the implementation of radial basis function generated finite differences (RBF-FD) for two-dimensional temperature diffusion over a plate. The comparison of the numerical results, traditional finite difference and exact solution are presented.

We obtained a satisfactory result when compared with finite difference solutions. The stabilized RBF-FD approach proved to be highly competitive against the traditional finite difference method.

Though the future of RBF-FD is optimistic, many topics still need to be addressed. These include dynamic adaptive node refinement, stability analysis of boundary conditions for hyperbolic problems, and the handling of discontinuities in the domain.

REFERENCES

1. Bollig, E.F., N. Flyer, and G. Erlebacher. 2012. "Solution to PDEs using Radial Basis Function Finite-Differences (RBF-FD) on Multiple GPUs". *Journal of Computational Physics.* 231:7133–7151.

2. Chinchapatnam, P.P., K. Djidjeli, P.B. Nair, and M. Tan. 2009. "A Compact RBF-FD Based Meshless Method for the Incompressible Navier-Stokes Equations". *Proc. IMechE, Part M- J. Eng. for Maritime Env*. 223: 275–290.

3. Driscoll, T.A. and B. Fornberg. 2002. "Interpolation in the Limit of Increasingly Flat Radial Basis Functions". *Comput. Math. Appl.* 43: 413–422.

4. Flyer, N., E. Lehto, S. Blaise, G.B. Wright, and A. St-Cyr. 2012. "A Guide to RBF-Generated Finite Differences for Nonlinear Transport: Shallow Water Simulations on a Sphere". *J. Comput. Phys.* 231: 4078–4095.

5. Fornberg, B. and G. Wright. 2004. "Stable Computation of Multiquadric Interpolants for all values of the Shape Parameter". *Comp. Math. with Applications*. 48:853-867.

6. Fornberg, B. and C. Piret. 2008. "On Choosing a Radial Basis Function and a Shape Parameter when Solving a Convective PDE on a Sphere". *J. Comput. Phys*. 227: 2758–2780.

7. Fornberg, B., G. Wright, and E. Larsson. 2004. "Some Observations Regarding Interpolants in the Limit of Flat Radial Basis Functions". *Comput. Math. Appl.* 47: 37–55.

8. Gonnet, P., R. Pachon, and L.N. Trefethen. 2011. "Robust Rational Interpolation and Least-Squares". *Electr. Trans. on Numer. Anal*. 33:146–147.

9. Larsson, E. and B. Fornberg. 2005. "Theoretical and Computational Aspects of Multivariate Interpolation with Increasingly Flat Radial Basis Functions". *Comput. Math. Appl*. 49: 103-130.

10. Schaback, R. 2005. "Multivariate Interpolation by Polynomials and Radial Basis Functions". *Constr. Approx.*. 21: 293–317.

11. Shan, Y.Y., C. Shu, and Z.L. Lu. 2008. "Application of Local MQ-DQ Method to Solve 3D Incompressible Viscous Flows with Curved Boundary". *Comp. Modeling in Eng. & Sci.* 25: 99–113.

12. Shu, C., H. Ding, and K.S. Yeo. 2003. "Local Radial Basis Function-Based Differential Quadrature Method and its Application to Solve Two Dimensional Incompressible Navier–Stokes Equations". *Comput. Methods Appl. Mech. Eng.* 192: 941–954.

13. Stevens, D., H. Power, M. Lees, and H. Morvan. 2009."The Use of PDE Centers in the Local RBF Hermitean Method for 3D Convective-Diffusion Problems". *J. Comput. Phys*. 228: 4606–4624.

14. Tolstykh, A.I. and D.A. Shirobokov. 2003. "On using Radial Basis Functions in a "Finite Difference Mode" with Applications to Elasticity Problems". *Comp. Mech.* 33:68–79.

15. Tolstykh, A.I. 2000. "On using RBF-Based Differencing Formulas for Unstructured and Mixed Structured-Unstructured Grid Calculations". *Proceedings of the 16th IMACS World Congress*: Lausanne.

16. Flyer, N. and B. Fornberg. 2011. "Radial Basis Functions: Developments and Applications to Planetary Scale Flows". *Computers and Fluids*. 26: 23-32.

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