Beta-Hjorth Distribution and its Properties

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ABSTRACT

This research shows the generalization of Beta and Hjorth distribution through their distribution functions and asymptotic properties. The resulting Beta-Hjorth Distribution (BHD) was defined and some of its properties like moment generating function, survival rate function, hazard rate function and cumulative distribution function were investigated. The distribution was found to generalize some known distributions with heavily tailed and skewed, thereby providing a great flexibility in modeling non-normal data.

(Keywords: Beta-Hjorth, hazard rate function, survival rate function, reliability)

INTRODUCTION

Development and Derivation of the Proposed Beta-Hjorth Distribution

Now by using the logit of beta defined by Jones, the mixture of Beta-Hjorth distribution can be obtained. Now let X be a random variable from the distribution with parameters and defined by eq(i) using the logit of Beta defined by Jones as:

$$g(x) = \frac{1}{B(a,b)}[G(x)]^{a-1}[1-G(x)]^{b-1}g(x) \quad \textbf{(1)}$$

where $g(x;\theta,\alpha) = (\theta+\alpha x)e^{-\left(\theta x+\frac{\alpha x^2}{2}\right)}$ and $G(x;\theta,\alpha) = 1-e^{-\left(\theta x+\frac{\alpha x^2}{2}\right)}$ are the pdf and cdf of Hjorth distribution, respectively. Then the Beta-Hjorth Distribution is obtained by using (1) are shown by (2).

$$f_{BHD}(x;\theta,\alpha,a,b) = \frac{1}{B(a,b)}\left[1-e^{-\left(\theta x+\frac{\alpha x^2}{2}\right)}\right]^{a-1}\left[e^{-\left(\theta x+\frac{\alpha x^2}{2}\right)}\right]^{b-1}(\theta+\alpha x)e^{-\left(\theta x+\frac{\alpha x^2}{2}\right)} \quad \textbf{(2)}$$

$$= \frac{1}{B(a,b)}\left[1-e^{-\left(\theta x+\frac{\alpha x^2}{2}\right)}\right]^{a-1}(\theta+\alpha x)\left[e^{-\left(\theta x+\frac{\alpha x^2}{2}\right)}\right]^{b-1}.e^{-\left(\theta x+\frac{\alpha x^2}{2}\right)} \quad \textbf{(3)}$$

$$f_{BHD}(x;\theta,\alpha,a,b) = \frac{(\theta+\alpha x)}{B(a,b)}\left[1-e^{-\left(\theta x+\frac{\alpha x^2}{2}\right)}\right]^{a-1}.\left[e^{-\left(\theta x+\frac{\alpha x^2}{2}\right)}\right]^{b} \quad \textbf{(4)}$$

The cdf can be obtained as follows:

$$F_{BHD}(x;\theta,\alpha,a,b) = \int_0^t \frac{(\theta+\alpha x)}{B(a,b)}\left[1-e^{-\left(\theta x+\frac{\alpha x^2}{2}\right)}\right]^{a-1}.\left[e^{-\left(\theta x+\frac{\alpha x^2}{2}\right)}\right]^{b}dx \quad \textbf{(5)}$$

$$= \frac{1}{B(a,b)} \int_0^t (\theta + \alpha x)\left[1 - e^{-\left(\theta x + \frac{\alpha x^2}{2}\right)}\right]^{a-1} \left[e^{-\left(\theta x + \frac{\alpha x^2}{2}\right)}\right]^{b} dx \quad \textbf{(6)}$$

Let y = $\theta x + \frac{\alpha x2}{2}$; $dx = \frac{dy}{(\theta+\alpha x)}$ put for $dx \ in \ (6)$

$$= \frac{1}{B(a,b)} \int_0^t (\theta + \alpha x)\left[1 - e^{-\left(\theta x + \frac{\alpha x^2}{2}\right)}\right]^{a-1} \left[e^{-\left(\theta x + \frac{\alpha x^2}{2}\right)}\right]^{b} . \frac{dy}{(\theta+\alpha x)} \quad \textbf{(7)}$$

$$= \frac{1}{B(a,b)} \int_0^t [1 - e^{-y}]^{a-1} [e^{-y}]^{b} \, \text{dy} \quad \textbf{(8)}$$

$$= \frac{1}{B(a,b)} \int_0^t M^{a-1} (1-M)^{b-1} \, dz \ ;$$

by following the approach of incomplete beta function, which states B(x;α,β) = $\int_0^t x^{\alpha-1} (1-x)^{\beta-1}$.

From this expression we can conclude that $\frac{1}{B(a,b)} \int_0^t M^{a-1} (1-M)^{b-1} \, dz = \frac{B(M;a,b)}{B(a,b)}$ = I_M(a, b) ,

where I_M(a, b) is the regularized incomplete beta function and M is the cdf of the parent distribution.

$$F_{BHD}(M) = I_Z(a, b) = \frac{B(M;a,b)}{B(a,b)} \quad \textbf{(9)}$$

$$= \frac{M^a}{B(a,b)} \sum_{n=0}^{\infty} \frac{(1-b)}{n!(a+n)} M^n \quad \textbf{(10)}$$

and M is the cdf of Hjorth. Therefore (4) and (10) are the pdf and cdf of Beta-Hjorth Distribution, respectively.

RELIABILITY

$$R(t) = \mathbf{1} - F_{BHD}(M) = 1 - \frac{B(M;a,b)}{B(a,b)} = \frac{B(a,b)-B(M;a,b)}{B(a,b)}$$

HAZARD FUNCTION

$$\mathbf{H(x)} = \frac{\frac{1}{B(a,b)}\left[1-e^{-\left(\theta x + \frac{\alpha x^2}{2}\right)}\right]^{a-1} (\theta+\alpha x)\left[e^{-\left(\theta x + \frac{\alpha x^2}{2}\right)}\right]^{b-1} .e^{-\left(\theta x + \frac{\alpha x^2}{2}\right)}}{\frac{B(a,b)-B(M;a,b)}{B(a,b)}}$$

$$= \frac{1}{B(a,b)}\left[1-e^{-\left(\theta x + \frac{\alpha x^2}{2}\right)}\right]^{a-1} (\theta+\alpha x)\left[e^{-\left(\theta x + \frac{\alpha x^2}{2}\right)}\right]^{b-1} .e^{-\left(\theta x + \frac{\alpha x^2}{2}\right)} \ x \ \frac{B(a,b)}{B(a,b)-B(M;a,b)}$$

$$= \frac{\left[1-e^{-\left(\theta x+\frac{\alpha x^2}{2}\right)}\right]^{a-1}(\theta+\alpha x)\left[e^{-\left(\theta x+\frac{\alpha x^2}{2}\right)}\right]^{b-1}.e^{-\left(\theta x+\frac{\alpha x^2}{2}\right)}}{B(a,b)-B(M;a,b)}$$

MOMENT

According to Cordeiro and DeCastro, the moment generating function of γ generated beta distribution is display as (9):

$$M_x(t) = \frac{1}{B(a,b)}\sum_{i=1}^{n}(-1)^i \binom{b-1}{i} \rho(t, ai-1) \quad \textbf{(9)}$$

Where $\rho(t,r) = \int_{-\infty}^{\infty} e^{tx}[F(x)]^r f(x)dx$ **(10)**

$$M_x(t) = \frac{1}{B(a,b)}\sum_{i=1}^{n}(-1)^i \binom{b-1}{i} \int_{-\infty}^{\infty} e^{tx}[F(x)]^r f(x)dx \quad \textbf{(11)}$$

$$M_x(t) = \frac{1}{B(a,b)}\sum_{i=1}^{n}(-1)^i \binom{b-1}{i} \int_{0}^{\infty} \left[1-e^{-\left(\theta x+\frac{\alpha x^2}{2}\right)}\right]^r (\theta+\alpha x)e^{-\left(\theta x+\frac{\alpha x^2}{2}\right)}dx \quad \textbf{(12)}$$

Let $= b = i = 1$, the mgf of Beta-Hjorth becomes the mgf of the parent distribution (Hjorth)

PARAMETER ESTIMATION (LOG-LIKELIHOOD)

Codeiro et al. (2011), stated (13) as the loglikelihood estimation method for estimating parameter $\beta = (a,b,c,\tau)$, where τ is the parameter vector of the parent distribution.

$$l(\theta) = nlogc - nlog[B(a,b)] + \sum_{i=1}^{n} logf(x_i,\tau) + (a-1)\sum_{i=1}^{n} logF(x_i,\tau) + (b-1)\sum_{i=1}^{n}\log(1-F(x_i,\tau)) \quad \textbf{(13)}$$

The generalized distribution reduces to the class of beta generated distribution when c=1, and the parameter vector $\beta\ now\ \beta = (a,b,\theta,\alpha)$.

By making necessary substitution, (13) becomes (14):

$$l(\beta) = -nlog[B(a,b)] + \sum_{i=1}^{n} log\left((\theta+\alpha x)e^{-\left(\theta x+\frac{\alpha x^2}{2}\right)}\right) + (a-1)\sum_{i=1}^{n} log\left(1-e^{-\left(\theta x+\frac{\alpha x^2}{2}\right)}\right) + (b-1)\sum_{i=1}^{n}\log(e^{-\left(\theta x+\frac{\alpha x^2}{2}\right)}) \quad \textbf{(14)}$$

$$\frac{\partial L(\beta)}{\partial a} = \frac{-n \acute{\Gamma}(a)}{\Gamma(a)} + \frac{\acute{\Gamma}(a+b)}{\Gamma(a+b)} + \sum_{i=1}^{n} log\left(1-e^{-\left(\theta x+\frac{\alpha x^2}{2}\right)}\right) \quad \textbf{(15)}$$

$$\frac{\partial L(\beta)}{\partial b} = \frac{-n \acute{\Gamma}(b)}{\Gamma(a)} + \frac{\acute{\Gamma}(a+b)}{\Gamma(a+b)} + \sum_{i=1}^{n}\log(e^{-\left(\theta x+\frac{\alpha x^2}{2}\right)}) \quad \textbf{(16)}$$

$$\frac{\partial L(\beta)}{\partial \theta} = \sum_{i=1}^{n} \frac{\partial}{\partial \theta} log\left((\theta + \alpha x)e^{-\left(\theta x + \frac{\alpha x^2}{2}\right)}\right) + (a-1)\sum_{i=1}^{n} \frac{\partial}{\partial \theta} log\left(1 - e^{-\left(\theta x + \frac{\alpha x^2}{2}\right)}\right) + (b - 1)\sum_{i=1}^{n} \frac{\partial}{\partial \theta} \log(e^{-\left(\theta x + \frac{\alpha x^2}{2}\right)}) \quad (17)$$

$$\frac{\partial L(\beta)}{\partial \theta} = \sum_{i=1}^{n} \frac{\partial}{\partial \theta} log\left((\theta + \alpha x)e^{-\left(\theta x + \frac{\alpha x^2}{2}\right)}\right) + (a-1)\sum_{i=1}^{n} \frac{\partial}{\partial \theta}\left(\left(\theta x + \frac{\alpha x^2}{2}\right)\right) + (b - 1)\sum_{i=1}^{n}\left(-\left(\theta x + \frac{\alpha x^2}{2}\right)\right) \quad (18)$$

$$\frac{\partial L(\beta)}{\partial \theta} = \sum_{i=1}^{n} \frac{1 - x\theta - \alpha x^2}{\alpha x + \theta} + (a-1)\sum_{i=1}^{n} x - (b-1)\sum_{i=1}^{n} x_i \quad (19)$$

$$\frac{\partial L(\beta)}{\partial \alpha} = \sum_{i=1}^{n} \frac{x(\alpha x^2 + \theta x - 2)}{2(x\alpha + \theta)} + (a-1)\sum_{i=1}^{n}\left(\frac{x^2}{2}\right) - (b-1)\sum_{i=1}^{n}\left(\frac{x^2}{2}\right) \quad (20)$$

The parameter $\hat{a}, \hat{b}, \hat{\alpha}\ and\ \hat{\theta}$ can be obtained via numerical method (Newton-Raphson Method) and necessary hypothesis can be tested using the fisher information matrix.

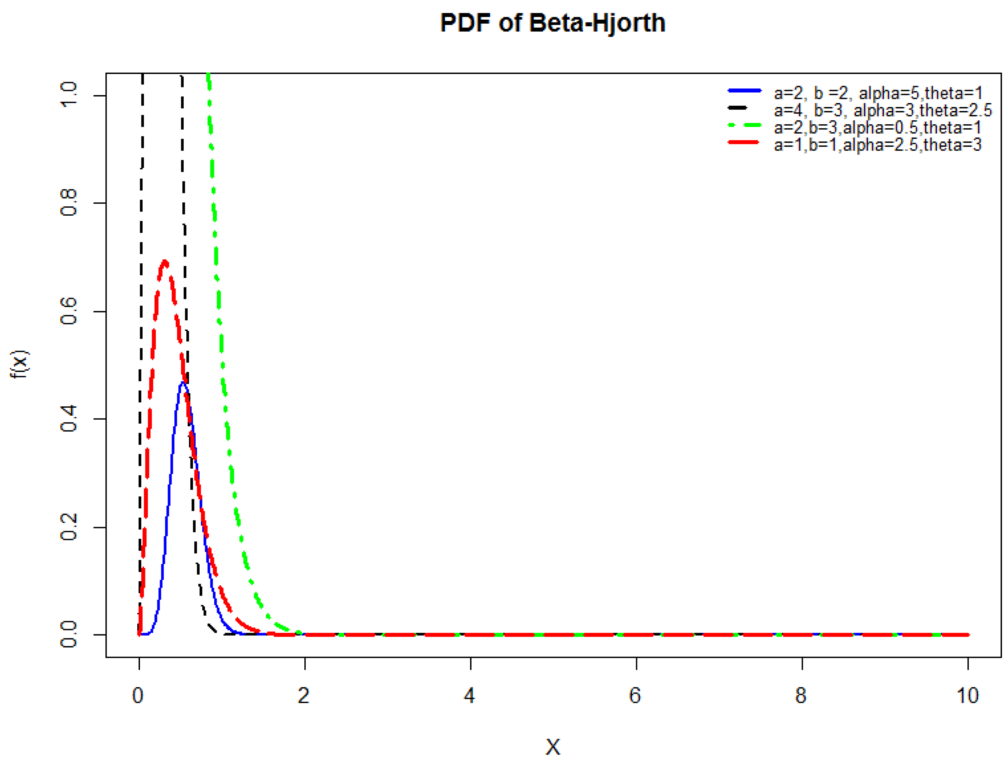

Figure 1: The PDF of Beta-Hjorth Distribution at Different Parameter Value.

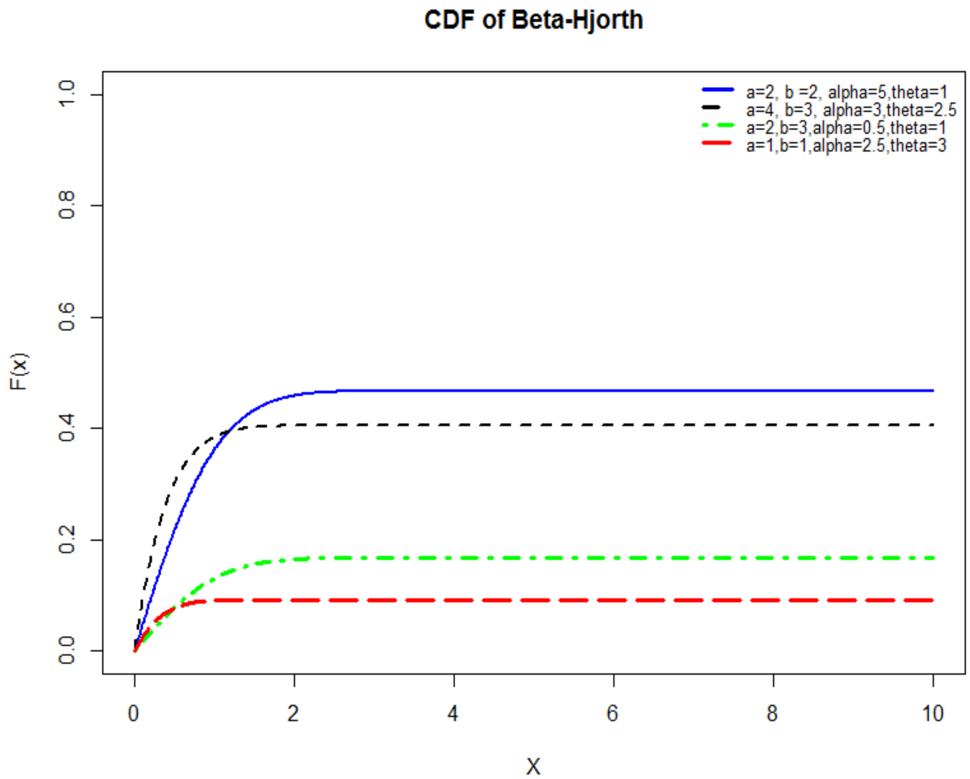

Figure 2: The CDF of Beta-Hjorth at Different Parameter Value.

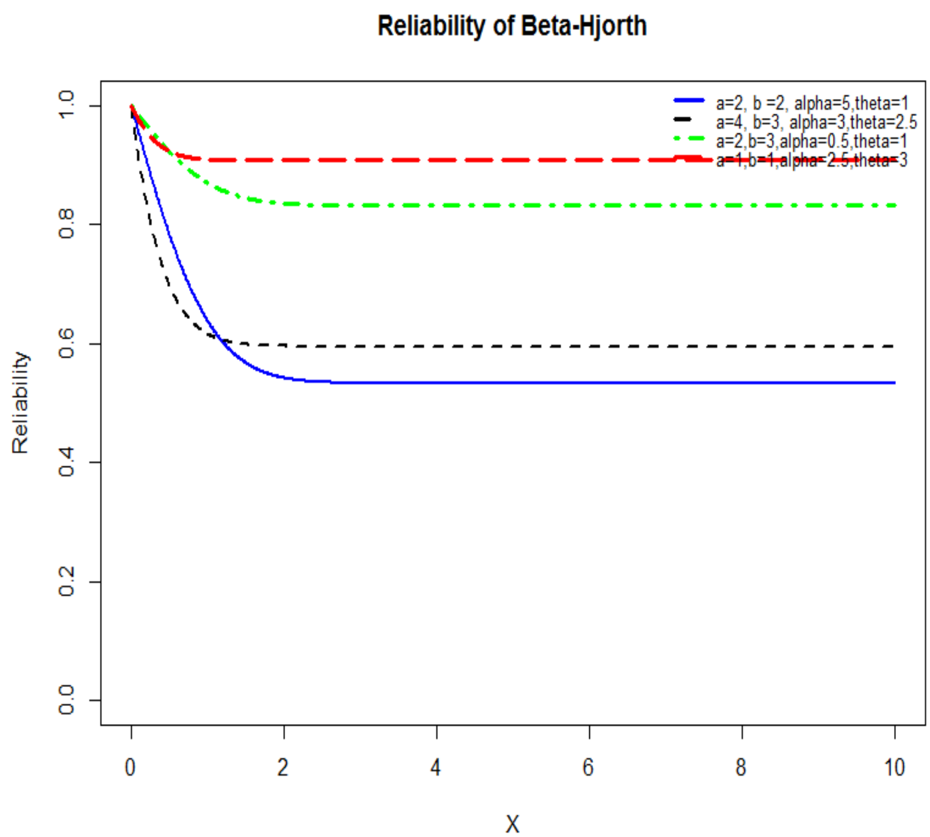

Figure 3: The Reliability of Beta-Hjorth at Different Parameter Value.

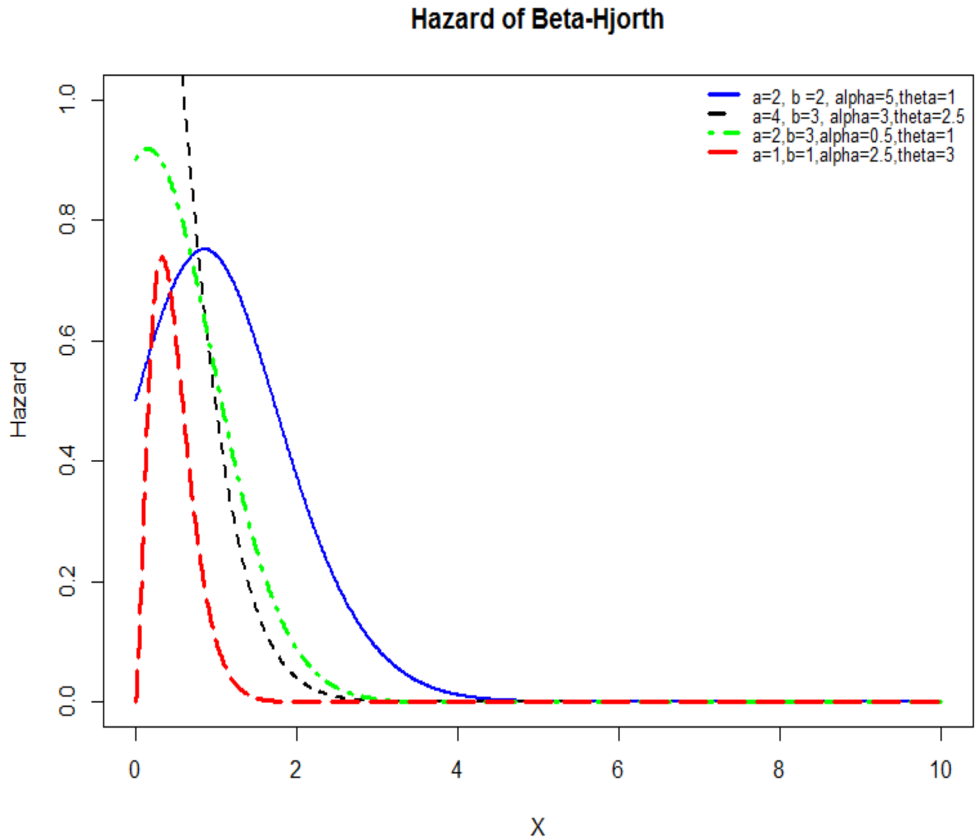

Figure 4: The Hazard Value of Beta-Hjorth at Different Parameter Values.

CONCLUSION

This work shows the mathematical derivation of the hybrid Beta-Hjorth and the parameter of this distribution can be obtained via numerical method preferably. Some mathematical properties along with estimation issues are addressed. We have presented an example where the mixed distribution fits better than the parent distribution. We believe that the subject distribution can be used in several different areas. We also believe this study will serve as a reference to advance future research in the subject area.

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