A Mathematical Study to Obtain a True Numerical Value of "e" used for Exponential Expression and as a Base for Napierian or Natural Logarithms

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ABSTRACT

In this paper, the binomial theorem has been used as a first-principle approach to obtain a true numerical value for a mathematical expression ''**e**''. Previous mathematical works established the value of e to be 2.71828. The proof was not from a first-principle approach, as it was based on another mathematical series for ''e'' itself. An in-depth study in this paper to establish the value of e, using the binomial theorem, gives a widely different value of e: 7.381847264.

(Keywords: mathematics, series, exponential expression, "e', logarithm, binomial theorem, infinity, limit)

INTRODUCTION

The lower case of the fifth letter in the alphabet, ''e'', was used by John Napier, a Scotsman, in 1614, to develop the natural logarithm (sometimes called Napierian logarithm) expressed as $\log_e x$ or simply ln x, pronounced ''leen x'' (Backhouse, et al., 1963). The letter ''e'' is equally used to express an exponent in the form e^x**.** A series had been developed for e^x, often referred to as the exponential series, and generally given by:

$$\boldsymbol{e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots} \quad -\infty < x < \infty$$

(Backhouse, et al, 1963) **(1)**

By substituting x=1 in (1), Backhouse, et al. (1963) showed that,

$$\boldsymbol{e^1 = e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + . \quad . \quad . + \frac{1}{n!} = 2.71828} \qquad \textbf{(2)}$$

For years, the expression shown in (2) has always served as a ''quick proof'' that e=2.71828. However, there are some mathematical facts behind the expression ''e'' and its numerical value. Until now, a first-principle approach for the establishment of a numerical value of ''e'', which is linked to the binomial theorem, has not heretofore been explored. These facts are all explored in this paper, and the results are hopefully informative.

<u>The Binomial Expansion and its Approach Towards Infinity</u>

In 1683, Jacob Bernoulli, a Swiss mathematician, was investigating the computation of compound interest (Biography of Jacob Bernoulli at en.m.wikipedia.org). In his work, Bernoulli attempted to evaluate the expression

$$\boldsymbol{\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n} \qquad \textbf{(3)}$$

He reported that the value must lie between 2 and 3. Bernoulli was however the first person to give an approximation for this limit, which is now referred to as "exponential number or "**e**". The number is often called Euler's number after Leonhard Euler, another Swiss mathematician who, we are told, approximated it to 18 decimal places (article at *www.mathscareers.org.uk*).

The expression in **(3)** is studied more closely using Table 1 below, compiled using Microsoft Excel®:

Table 1: A Tabulation of (Equation 3) as n varies from 10^0 to 10^{11}

n	$y = (1 + \frac{1}{n})^n$
10^0	2
10^1	2.59374246
10^2	2.704813829
10^3	2.716923932
10^4	2.718145927
10^5	2.718268237
10^6	2.718280469
10^7	2.718281694
10^8	2.718281786
10^9	2.718282031
10^{10}	2.718282053
10^{11}	2.718282053

The Scientific Community has adopted 2.71828 (5 decimal places) as the value of e= $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$.

The authors of this paper suspect that the reason for this is that 2.71828 holds for a wide range of values of **n**, from 10^6 to 10^{11} as shown in table 1 above. From the table, it was observed that as **n** increased, **y** also increased. The graph of $y = (1+\frac{1}{n})^n$ in Figure 1 below is an overview of the variation within the range of practical values of **n**.

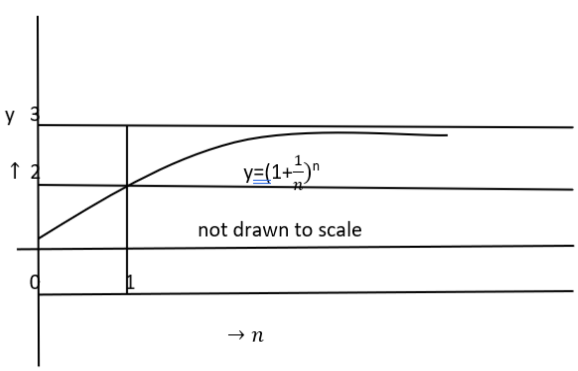

Figure 1: A Sketch of (Equation 3) within the Range of Practical Values of n.

For values of n, which can practically be shown on graph paper, the value of y approached the number 3, as suggested by Bernoulli. This is equally illustrated in Figure 1. However, out of sheer curiosity, the authors of this paper decided to investigate the outcome if the value of n was increased, as shown in Table 2 below. What was eventually obtained for $(1+\frac{1}{n})^n$ was worrisome.

Table 2: A Tabulation of (Equation 3) to Show how $(1+\frac{1}{n})^n$ Varies for Higher Values of n.

n	$y = (1 + \frac{1}{n})^n$
10^{12}	2.718523496
10^{13}	2.716110034
10^{14}	2.716110034
10^{15}	3.035035207
10^{16}	1
10^{17}	1
10^{18}	1

The following salient, pertinent, and intuitive questions then came to mind for the authors: why should **y** decrease between **n**=10^{12} and **n**=10^{14}, then increase when **n** got to 10^{15}, only to finally settle down at **1** for higher values of **n**? They finally decided to try values of **n** between 10^{15} and 10^{16} which, of course would not be integral powers of 10, and obtained shocking results. It was amazing that when **n=9002803354665479**, the computer gave $(1+\frac{1}{n})^n$=**7.381847264.** The difference between this result and 2.71828 was too wide to be ignored.

Again, the salient question that came to the mind of the authors was: ''was the computer wrong?'' It was then that an exploration into the binomial expansion of $(1+\frac{1}{n})^n$ was undertaken, to search for a possible explanation (Itaketo, 2015).

First, the following were considered: 1/1=1, 1/2=0.5, 1/3=0.33…, 1/4=0.25, and so on. It was observed that as the denominator increased, the value of the fraction decreased, hence it was safely concluded that $\frac{1}{\infty} = 0$.

Secondly, taking a close look at the binomial expansion of $(1+\frac{1}{n})^n$ for positive integral values of **n**, it was observed that:

$$(1+\frac{1}{n})^n=1+{}^nC_1(\frac{1}{n})+{}^nC_2(\frac{1}{n})^2+{}^nC_3(\frac{1}{n})^3+\ldots+(\frac{1}{n})^n \quad \textbf{(4)}$$

$$=1+n(\frac{1}{n})+{}^{n}C_2(\frac{1}{n})^2+{}^{n}C_3(\frac{1}{n})^3+\ldots+(\frac{1}{n})^n \quad \textbf{(5)}$$

$$=1+1+{}^{n}C_2(\frac{1}{n})^2+{}^{n}C_3(\frac{1}{n})^3+\ldots+(\frac{1}{n})^n \quad \textbf{(6)}$$

$$=1+1+({}^{n}P_2/2!)(\frac{1}{n})^2+({}^{n}P_3/3!)(\frac{1}{n})^3+\ldots+(\frac{1}{n})^n \quad \textbf{(7)}$$

$$=1+1+\frac{n(n-1)}{2n^2}+\frac{n(n-1)(n-2)}{6n^3}+\ldots+(\frac{1}{n})^n \quad \textbf{(8)}$$

Now, again, taking a close look at the third term in (8), that term could be expressed as:

$$\frac{n(n-1)}{2n^2}=\frac{n}{2n}\times\frac{n-1}{n}$$

Clearly, the third term in (8) is a positive proper fraction!

DISCUSSION

In an attempt to explain how and why the computer gave $(1+\frac{1}{n})^n=1$ when $n=10^{16}$ and above (precisely when n = 9002803354665480 and above), it was seen earlier at (5) above, that $(1+\frac{1}{n})^n= 1+n(\frac{1}{n})+{}^{n}C_2(\frac{1}{n})^2+{}^{n}C_3(\frac{1}{n})^3+\ldots+(\frac{1}{n})^n$.

Now, beginning from the second term in the expansion, right up to the last term, the fraction $\frac{1}{n}$ appears. Considering that $\frac{1}{\infty}$ = 0, as shown earlier, it is suspected that the computer approximated every $\frac{1}{n}$ in (5) above and made it equal to 0 when **n** became too large for it to handle. What is being postulated here, in other words, is that the result $(1+\frac{1}{n})^n=1$ arose from an unfortunate approximation by the computer. After all, infinity, ∞, is a truly transcendental (pun - intended) number.

CONCLUSION

It is realized, in like manner, that all the other terms from the fourth to the last in (8) above, are positive proper fractions. Without further ado, it could safely be concluded that for $1< n < \infty$, $(1+\frac{1}{n})^n>2$, and that $(1+\frac{1}{n})^n$ increases as n increases within this range. This would probably explain why the computer gave greater values than **2.71828** for $(1+\frac{1}{n})^n$. The highest value obtained was **7.381847264** as mentioned earlier.

As stated above, the difference between 7.381847264 and 2.71828 was, and is, too wide to be ignored. The binomial theorem has really served as a firm basis for the investigation.

Now, the question for the mathematics and scientific communities in the world is: What really is the true value of e, **2.71828** or **7.381847264**? Where do we go from here? The mathematics and scientific communities probably need to rethink the true value of **e**.

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