An Alternative Solution to Hotelling T Square under the Heteroscedasticity of the Dispersion Matrix

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ABSTRACT

This work focuses on developing an alternative procedure to multivariate Behrens–Fisher problem by using approximate degree of freedom test which was adopted from Satterthwaite univariate procedure. The proposed procedure was compared via R package simulation and real-life data used by Timm with six (6) existing procedures namely: Johanson, Yao, Krishnamoorthy, Hotelling T square, Nel and Van der Merwe, and Yanagihara. It was discovered that the proposed procedure performed better in terms of power of the test, then all existing procedures considered in all the scenarios that include at different: (i) random variables (p), (ii) variance co–variance matrix, (iii) sample size, and (iv) significance level (α). The proposed procedure is completely favorably, well in terms of type I error rate with Johanson, Yao, Krishnamoorthy, Nel, and Van der Merwe.

(Keywords: variance, co-variance matrix, linear combination, type 1 error rate, power the test, heteroscedasticity, R statistical package)

INTRODUCTION

Suppose we have a random sample of size n_1, $x_{11}, x_{12}, x_{13}, \ldots x_{1n_1}$ for $N(\mu_1, \sigma_1^2)$ and a second random sample of size n_2, $x_{21}, x_{22}, x_{23}, \ldots x_{2n_2}$ for $N(\mu_2, \sigma_2^2)$. It is desired to test $H_0: u_1 = u_2$ against $H_1: u_1 \neq u_2$. If σ_1 and σ_2 are both known a normal test is used. If $\sigma_1 = \sigma_2$ but both are unknown a t-test is commonly used with the test statistics.

$$t = \frac{(\bar{x}_1 - \bar{x}_2)}{S_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}}$$

Where $S_p^2 = \frac{(n_1-1)S_1^2 + (n_2-1)S_2^2}{n_1+n_2-2}$,

$$\bar{x}_i = \frac{1}{n_i}\sum x_{ij}$$

and

$$S_i^2 = \frac{\sum(x_{ij}-\bar{x}_i)^2}{n_i-1} \quad i = 1, 2$$

A t test is a type of statistical test that is used to compare the means of two groups, such as men vs. women, athletes vs. non-athletes, young vs. elderly, or you may want to compare means measured on a single group under two different experimental conditions or at two different times.

T tests are a type of parametric method; they can be used when the samples satisfy the conditions of normality, equal variance, and independence. T tests can be divided into two types. There is the independent t test, which can be used when the two groups under comparison are independent of each other, and the paired t test, which can be used when the two groups under comparison are dependent on each other. If $\sigma_1 \neq \sigma_2$ and both are unknown, then we are confronted with the Behrens-Fisher problem. There is no universally accepted testing procedure for this problem although arrays of tests have been developed and will be discussed in the Review of Literature. Behrens (1929) proposed the statistics.

In the literature, there are modified test statistics (*t* test) when the assumption of equal variances is violated and has been known as the Behrens-Fisher problem (Behrens, 1929; Fisher, 1935). Early investigations showed that the problem can be overcome by substituting separate-variances tests, such as the ones introduced by Welch (1938, 1947), and Satterthwaite (1946), for the Student *t* test. These modified significance tests,

unlike the usual two-sample Student t test, do not pool variances in computation of an error term. Moreover, they alter the degrees of freedom by a function that depends on sample data.

It has been found that these procedures in many cases restore Type I error probabilities to the nominal significance level, and also, counteract increase or decrease of Type II error probabilities (see, for example, Overall, Atlas, and Gibson, 1995a, 1995b; Zimmerman, 2004; Zimmerman and Zumbo, 1993).

Student's t test is univariate and analogue to Hotelling T square which is the multivariate version of T – test and this Hotelling's T^2, has three basic assumptions that are fundamental to the statistical theory: independent, multivariate normality, and equality of variance-covariance matrices. A statistical test procedure is said to be robust or insensitive if departures from these assumptions do not greatly affect the significance level or power of the test.

To use Hotelling's T^2 one must assume that the two samples are independent and that their variance-covariance matrices are equal $(\Sigma_1 = \Sigma_2 = \Sigma)$. When variance–covariance matrices are not homogeneous and unknown, the test statistic will not be distributed as a T^2. This predicament is known as the multivariate Behrens-Fisher problem.

The Behrens-Fisher Problem addresses interval estimation and hypothesis testing concerning the differences between the means of two normally distributed populations when the variances of the two populations are not equal. While Multivariate Behrens-Fisher problem deal with testing the equality of two normal mean vector under heteroscedasticity of dispersion matrices. These are the some of the existing Multivariate Behrens–Fisher problem as demonstrated by:: Yao (1965), Johanson (1980), Nel et al. (1990), Kim (1992), Krishnamoorthy and Yu (2004), Gamage et al (2004), Yanagihara and Yuan (2005), Kawasaki and Seo (2012) and so on. But with all these procedures there is not one that is one hundred percent (100%) satisfactory in terms of power of the test and type I error rate.

Each of these scholar works on the degree of freedom using different method which are classified into four (4): approximate degree of freedom tests, series expansion-based tests, simulation-based tests, and transformation-based tests.

Scheffe (1970), Lauer and Han (1974), Lee and Gurland (1975), Timm (1975), Murphy (1976), Yao (1965), Algina and Tang (1988), Kim (1992), De la Rey and Nel's (1993), Christensen and Rencher (1997), and Oyeyemi and Adebayo (2016). All these authors mentioned and many more have work on comparison of some of the Multivariate Behrens–Fisher problem procedures.

The purpose of this work is to develop an alternative procedure for multivariate data that will be robust compared to other procedures and the work will begin with an introduction to the statistical notation that will be helpful in understanding the concepts. This is followed by a discussion of procedures that can be used to test the hypothesis of multivariate mean equality when statistical assumptions are and are not satisfied. We will then show how to obtain a test that is robust to the covariance heterogeneity.

Multivariate Behrens–Fisher Problem (existing procedure)

Consider two p–variate normal populations $N\left(\underline{\mu}_1, \Sigma_1\right)$ and $N\left(\underline{\mu}_2, \Sigma_2\right)$ where $\underline{\mu}_1$ and $\underline{\mu}_2$ are unknown $p \times 1$ vectors and Σ_1 and Σ_2 are unknown $p \times p$ positive definite matrices. Let $X_{\alpha 1} \sim N\left(\underline{\mu}_1, \Sigma_1\right), \alpha = 1, 2, \ldots, n_1$,

and

$$X_{\alpha 2} \sim N\left(\underline{\mu}_2, \Sigma_2\right), \alpha = 1, 2, \ldots, n_2,$$

denote random samples from these two populations, respectively. We are interested in the testing problem:

$$H_o : \underline{\mu}_1 = \underline{\mu}_2 \quad vs \quad H_1 : \underline{\mu}_1 \neq \underline{\mu}_2$$

For $i = 1, 2,$ let

$$\bar{X}_i = \frac{1}{n_i} \sum_{\alpha=1}^{n_i} X_{\alpha i}$$

$$A_i = \sum_{n=1}^{n_i} (X_{\alpha i} - \bar{X}_i)(X_{\alpha i} - \bar{X}_i)'$$

$$S_i = A_i / (n_i - 1), \quad i = 1, 2.$$

Then $\bar{X}_1, \bar{X}_2, A_1$ and A_2, which are sufficient for the mean vectors and dispersion matrices, are independent random variables having the distributions:

$$\bar{X}_i \sim N\left(\mu_i, \frac{\Sigma_i}{n_i}\right), and \; A_i \sim W_p(n_i - 1, \Sigma_i), i = 1, 2$$

Where $W_p(r, \Sigma)$ denotes the p - dimensional Wishart distribution with $df = r$ and scale matrix Σ.

$$\tilde{S}_i = S_i/n_i, \;\; i = 1, 2.$$

$$\tilde{S} = \tilde{S}_1 + \tilde{S}_2$$

$$T^2 = (\bar{X}_1 - \bar{X}_2)'\tilde{S}^{-1}(\bar{X}_1 - \bar{X}_2)$$

The following are the existing procedures or solutions to Multivariate Behrens-Fisher problem considered in this study

1. Yao Procedure

Yao (1965)'s, Ajit and Brent (2002) invariant test. This is a multivariate extension of the Welch 'approximate degree of freedom' solution provided by Turkey and the test statistic is based on a transformation of Hotelling T^2. And is based on $T^2 \sim (vp/(v-p+1))F_{p,v-p+1}$ with the degrees of freedom v given by:

$$\frac{1}{v} = \frac{1}{(T^2)^2}\sum_{i=1}^{2}\frac{1}{n_i - 1}\left[(\bar{x}_1 - \bar{x}_2)'\tilde{S}^{-1}\tilde{S}_i\tilde{S}^{-1}(\bar{x}_1 - \bar{x}_2)\right]^2$$

$$T_{Yao} = \frac{(v-p+1)T^2}{pv}$$

Statistical significance is then assessed by comparing the T_{Yao} statistic to its critical value $F_\alpha(p, \; v-p+1)$, that is, a critical value from the F distribution with p and $v - p + 1$ degrees of freedom (df)

2. Johansen Procedure

Johansen (1980)'s, invariant test, Yanagihara and Yuan (2005), Kawasaki and Takashi. (2011). They used $T^2 \sim qF_{pv}$ where:

$$q = p + 2D - 6D/[p(p-1) + 2],$$

$$v_{Joh} = p(p+2)/3D$$

$$D = \frac{1}{2}\sum_{i=1}^{2}\left\{tr\left[(I - (\tilde{S}_1^{-1} + \tilde{S}_2^{-1})^{-1}\tilde{S}_i^{-1})^2\right] + tr\left[(I - (\tilde{S}_1^{-1} + \tilde{S}_2^{-1})^{-1}\tilde{S}_i^{-1})\right]^2\right\}/n_i$$

And his proposed test statistic:

$$T_{Johan} = \frac{T^2}{q}$$

Statistical significance is then assessed by comparing the T_{Johan} statistic to its critical value $F_\alpha(p, \; v_{Joh})$, that is, a critical value from the F distribution with p and v_{Joh} degrees of freedom (df)

3. Nel and Van der Merwe

Nel and Van der Merwe (1986) noninvariant solution. Here we use:

$$T^2 \sim (v_{Nv}p/(v_{Nv} - p + 1))F_{p,v_{Nv}-p+1} \text{ except that } v \text{ is defined by:}$$

$$v_{Nv} = \frac{tr(S^2)+[tr(S)]^2}{\left(\frac{1}{n_1}\right)\{tr(S_1^2)+[tr(S_1)]^2\}+\left(\frac{1}{n_2}\right)\{tr(S_2^2)+[tr(S_2)]^2\}}$$

4. Krishnamoorthy and Yu Procedure

Krishnamoorthy and Yu (2004)'s, Lin and Wang (2009), modified Nel/ Van der Merwe invariant solution. We use the idea as before, namely,

$$T^2 \sim (v_{kY}p/(v_{kY} - p + 1))F_{p,v_{kY}-p+1} \text{ with the } d.f.v \text{ defined by:}$$

$$v_{KY} = (p + p^2)/C$$

$$C = \frac{1}{n_1}\left\{tr\left[(\tilde{S}_1\tilde{S}^{-1})^2\right] + [tr(\tilde{S}_1\tilde{S}^{-1})]^2\right\} + \frac{1}{n_2}\left\{tr\left[(\tilde{S}_2\tilde{S}^{-1})^2\right] + [tr(\tilde{S}_2\tilde{S}^{-1})]^2\right\}$$

$$T_{krish} = \frac{(v_{kY}-p+1)T^2}{pv_{KY}}$$

Statistical significance is then assessed by comparing the T_{Krish} statistic to its critical value $F_\alpha(p, \; v_{kY} - p + 1)$, that is, a critical value from the F distribution with p and $v_{kY} - p + 1$ degrees of freedom (df).

5. Yanagihara and Yuan Procedure

Yanagihara and Yuan Procedure (2005) used series expansion-based test to develop an alternative procedure to Multivariate Behrens – Fisher problem:

$$T_{Yan} = \frac{n - 2 - \hat{\theta}_1}{(n - 2)p} T \sim F_{p,\hat{v}_{yan}}$$

$$\hat{v}_{yan} = \frac{(n - 2 - \hat{\theta}_1)^2}{(n - 2)\hat{\theta}_2 - \hat{\theta}_1}$$

$$\hat{\theta}_1 = \frac{p\hat{\psi}_1 + (p - 2)\hat{\psi}_2}{p(p + 2)} \quad and \quad \hat{\theta}_2 = \frac{\hat{\psi}_1 + 2\hat{\psi}_2}{p(p + 2)}$$

$$\hat{\psi}_1 = \frac{n_2^2(n - 2)}{n^2(n_1 - 1)}\left\{tr\left(S_1\bar{S}^{-1}\right)\right\}^2 + \frac{n_1^2(n - 2)}{n^2(n_2 - 1)}\left\{tr\left(S_2\bar{S}^{-1}\right)\right\}^2$$

$$\hat{\psi}_2 = \frac{n_2^2(n-2)}{n^2(n_1-1)} tr\left(S_1\bar{S}^{-1}S_1\bar{S}^{-1}\right) + \frac{n_1^2(n-2)}{n^2(n_2-1)} tr\left(S_2\bar{S}^{-1}S_2\bar{S}^{-1}\right)$$

$$\bar{S} = \frac{n_2}{n}S_1 + \frac{n_1}{n}S_2$$

6. <u>Hotelling's T^2</u>

$$T_1 = \frac{n_1 n_2}{n_1+n_2}(\bar{x}_1 - \bar{x}_2)'S_{pl}^{-1}(\bar{x}_1 - \bar{x}_2)$$

Where,

$$S_{pl} = \frac{1}{n_1+n_2-2}[(n_1-1)S_1 + (n_2-1)S_2]0$$

The test statistic can also be converted to an F statistic,

$$v_{hotel} = \frac{N-p-1}{p(N-2)}T_1$$

Where $N = n_1 + n_2$. Statistical significance is then assessed by comparing the v_{hotel} statistic to its critical value $F_\alpha(p, N-p-1)$, that is, a critical value from the F distribution with p and $N-p-1$ degrees of freedom (df).

The Mean and Variance of the Chi – square Distribution with n Degrees of Freedom

The chi – square distribution is defined with n degrees of freedom by $\chi_n^2 = Z_1^2 + Z_2^2 + \ldots + Z_n^2$, where $Z_1, Z_2, \ldots Z_n$ are independent random variables, each with distribution $N(0, 1)$.

Find the expected value and variance of both sides, then we have:

$$E(\chi_n^2) = E(Z_1^2) + \ldots + E(Z_n^2) \text{ and}$$

$$Var(\chi_n^2) = Var(Z_1^2) + \ldots + Var(Z_n^2)$$

And all the instances of Z_i have identical distributions, then:

$$E(\chi_n^2) = nE(Z^2) \text{ and}$$

$$Var(\chi_n^2) = nVar(Z^2)$$

Where Z is the random variable with distribution $N(0, 1)$. Then $E(Z^2) = E[(Z-0)^2] = E[(Z-u_z)^2] = Var(Z) = 1$

Therefore:

$$E(\chi_n^2) = n.1 = n \qquad\qquad\qquad (1)$$

For $Var(Z^2)$

$$Var(Z^2) = E[(Z - \mu_{z^2})^2] = E[(Z^2 - 1)^2] = E(Z^4 - 2Z^2 + 1)$$
$$= E(Z^4) - 2E(Z^2) + 1 = E(Z^4) - 2.1 + 1$$

Now,

$$Var(Z^2) = E(Z^4) - 1 \tag{2}$$

To find $E(Z^4)$, we will use the fact that for any continuous random variable X with probability density function f, and any exponent k,

$$E(X^k) = \int_{-\infty}^{\infty} x^k f(x) dx$$

And that the probability density function f of the $N(0, 1)$ random variable is given by:

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$

Then, $E(Z^4) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^4 e^{-\frac{z^2}{2}} dz$

By integration by parts, we have, $E(Z^4) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^3 . ze^{-\frac{z^2}{2}} dz$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \left[z^3 . \left(-e^{-\frac{z^2}{2}} \right) \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} 3z^2 \left(-e^{-\frac{z^2}{2}} \right) dz \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ 0 + \int_{-\infty}^{\infty} 3z^2 e^{-\frac{z^2}{2}} dz \right\}$$

$$= 3 . \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2 e^{-\frac{z^2}{2}} dz$$

$$= 3E(z^2) = 3.1 = 3 \tag{3}$$

Therefore, substituting Equation (3) into Equation (2) then we have:

$$Var(Z^2) = 3 - 1 = 2, \text{ that is } Var(\chi_n^2) = n.2 = 2n \tag{4}$$

For two sample t – test, we will limit this work to the version of the test where we do not assume that the two populations have equal variances. Let random sample $x_1, \ldots, x_{n1}$ from a random variable X with distribution $N(\mu_1, \sigma_1)$ and a random sample $y_1, \ldots, y_{n2}$ from a random variable Y with distribution (μ_2, σ_2) . we have:

$$t = \frac{(\bar{X} - \bar{Y}) - (\mu_1 - \mu_2)}{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}}$$

Strictly speaking, this statistic does not follow t – distribution, therefore:

The variance of $\bar{X} - \bar{Y}$ is $\sigma_B^2 = \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}$ and as an estimator for σ_B^2 we have $s_B^2 = \frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}$

For t to be t – distribution, there would have to be some multiple of s_B^2 that is chi – square distribution and this is not the case. However, remember that in the one – sample case, $\frac{(n-1)s^2}{\sigma^2}$ had a chi – square distribution with *n- 1* degrees of freedom. By analogy, we have $\frac{rs_B^2}{\sigma_B^2}$ has a chi – square distribution with r degrees of freedom. Satterthwaite found the true distribution of s_B^2 and showed that if *r* is chosen so that the variance of the chi – square distribution with r degrees of freedom is equal to the true variance of $\frac{rs_B^2}{\sigma_B^2}$, then, under certain conditions, this chi – square distribution with *r* degrees of freedom is a good approximation to be true distribution of $\frac{rs_B^2}{\sigma_B^2}$. so from this point, we are assuming that $\frac{rs_B^2}{\sigma_B^2}$ has distribution χ_r^2. So from Equation (4):

$$Var\left(\frac{rs_B^2}{\sigma_B^2}\right) = 2r \qquad (5)$$

$$Var\left(\frac{rs_B^2}{\sigma_B^2}\right) = \frac{r^2}{\sigma_B^4}Var(s_B^2) \qquad (6)$$

Solving the Equations (5) and (6):

$$2r = \frac{r^2}{\sigma_B^4}Var(s_B^2)$$

$$\frac{2}{r} = \frac{1}{\sigma_B^4}Var(s_B^2) \qquad (7)$$

Now $s_B^2 = \frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}$, and s_1 and s_2 are independent so,

$$Var(s_B^2) = \frac{1}{n_1^2}Var(s_1^2) + \frac{1}{n_2^2}Var(s_2^2) \qquad (8)$$

We know that $\frac{(n_1-1)s_1^2}{\sigma_1^2}$ has a chi – square distribution with $n_1 - 1$ degrees of freedom, from Equation (3):

$$Var\left[\frac{(n_1-1)s_1^2}{\sigma_1^2}\right] = 2(n_1 - 1)$$

$$\frac{(n_1-1)^2}{\sigma_1^2}Var(s_1^2) = 2(n_1 - 1)$$

$$Var(s_1^2) = \frac{2\sigma_1^4}{n_1-1} \quad \text{and similarly,} \quad Var(s_2^2) = \frac{2\sigma_2^4}{n_2-1} \qquad (9)$$

Substituting Equation (9) into Equation (8):

$$Var(s_B^2) = \frac{1}{n_1^2}\cdot\frac{2\sigma_1^4}{n_1-1} + \frac{1}{n_2^2}\cdot\frac{2\sigma_2^4}{n_2-1} \qquad (10)$$

Substituting Equation (10) into Equation (7):

$$\frac{2}{r} = \frac{1}{\sigma_B^4} \left(\frac{1}{n_1^2} \cdot \frac{2\sigma_1^4}{n_1 - 1} + \frac{1}{n_2^2} \cdot \frac{2\sigma_2^4}{n_2 - 1} \right)$$

$$r = \frac{\sigma_B^2}{\frac{1}{n_1^2} \cdot \frac{2\sigma_1^4}{n_1 - 1} + \frac{1}{n_2^2} \cdot \frac{2\sigma_2^4}{n_2 - 1}} \tag{11}$$

In practice, the values of the population variances, σ_1^2 and σ_2^2, are unknown, and so we replace σ_1^2, σ_2^2, and σ_R^2 by their estimators s_1^2, s_1^2 and s_R^2. also $s_B^2 = \frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}$ from Equation (11):

$$r = \frac{s_B^2}{\frac{1}{n_1^2} \cdot \frac{2s_1^4}{n_1 - 1} + \frac{1}{n_2^2} \cdot \frac{2s_2^4}{n_2 - 1}} = \frac{\left(\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2} \right)^2}{\frac{1}{n_1 - 1} \cdot \left(\frac{S_1^2}{n_1} \right)^2 + \frac{1}{n_2 - 1} \cdot \left(\frac{S_2^2}{n_2} \right)^2}$$

Multivariate Approach

We shall consider the test statistic $y'S^{-1}y$ and use Univariate Satterthwaite approximation of degrees of freedom method to suggest multivariate generalization based on the T^2 – distribution. We have:

$$S = \hat{S}_1 + \hat{S}_2, \qquad \hat{S}_i = \frac{S_i}{n_i} \qquad \text{and} \quad y = \bar{X}_1 - \bar{X}_2$$

$$y \sim N(0, \Sigma)$$

$$b'y \sim N(0, b'\Sigma b)$$

Where b is an arbitrary constant vector:

$$v(b'Sb) \sim (b'\Sigma b)\chi^2_{(v)}$$

$$\frac{v(b'Sb)}{(b'\Sigma b)} \sim \chi^2_{(v)} \qquad \text{Multivariate of version of Equation (5)}$$

$$Var\left(\frac{v(b'Sb)}{(b'\Sigma b)} \right) = 2v \tag{12}$$

Multivariate of version of Equation (5) is Equation (12):

$$\frac{v^2}{(b'\Sigma b)^2} Var(b'Sb) = 2v$$

$$Var(b'Sb) = \frac{2v(b'\Sigma b)^2}{v^2}$$

$$\frac{2(b'\Sigma b)^2}{v} = Var(b'Sb)$$

$$\frac{2}{v} = \frac{1}{(b'\Sigma b)^2} Var(b'Sb) \tag{13}$$

Note $S = \frac{S_i}{n_i} + \frac{S_2}{n_2}$ that is $(b'Sb) = \frac{(b'S_1 b)}{n_i} + \frac{(b'S_2 b)}{n_2}$ and $(b'\Sigma b) = \frac{(b'\Sigma_1 b)}{n_i} + \frac{(b'\Sigma_2 b)}{n_2}$ **(14)**

Putting Equation (14) into Equation (13):

$$\frac{2}{v} = \frac{1}{(b'\Sigma b)^2} Var\left(\frac{(b'S_1 b)}{n_i} + \frac{(b'S_2 b)}{n_2} \right)$$

$$\frac{2}{v} = \frac{1}{(b'\Sigma b)^2} Var\left(\frac{1}{n_1^2}.Var(b'S_1 b) + \frac{1}{n_2^2}.Var(b'S_2 b) \right) \qquad \textbf{(15)}$$

Multivariate of version of Equation (9) is:

$$Var(b'S_1 b) = \frac{2\big((b'\Sigma_1 b)\big)^2}{n_1 - 1} \quad \text{and} \quad Var(b'S_2 b) = \frac{2\big((b'\Sigma_2 b)\big)^2}{n_2 - 1} \qquad \textbf{(16)}$$

Placing Equation (16) into Equation (15):

$$\frac{2}{v} = \frac{1}{(b'\Sigma b)^2}\left[\frac{1}{n_1^2}.\frac{2\big((b'\Sigma_1 b)\big)^2}{n_1 - 1} + \frac{1}{n_2^2}.\frac{2\big((b'\Sigma_2 b)\big)^2}{n_2 - 1} \right]$$

$$\frac{2}{v} = \frac{2}{(b'\Sigma b)^2}\left[\frac{1}{n_1^2}.\frac{\big((b'\Sigma_1 b)\big)^2}{n_1 - 1} + \frac{1}{n_2^2}.\frac{\big((b'\Sigma_2 b)\big)^2}{n_2 - 1} \right]$$

$$\frac{1}{v} = \frac{1}{(b'\Sigma b)^2}\left[\frac{1}{n_1^2}.\frac{\big((b'\Sigma_1 b)\big)^2}{n_1 - 1} + \frac{1}{n_2^2}.\frac{\big((b'\Sigma_2 b)\big)^2}{n_2 - 1} \right] \qquad \textbf{(17)}$$

Equation (17) becomes:

$$v = \frac{(b'\Sigma b)^2}{\frac{1}{n_1^2}.\frac{\big((b'\Sigma_1 b)\big)^2}{n_1 - 1} + \frac{1}{n_2^2}.\frac{\big((b'\Sigma_2 b)\big)^2}{n_2 - 1}}$$

The values of the population variances Σ_i are unknown, and so we replace Σ_i and $b'\Sigma b$ by their estimators S_i and $b'Sb$:

$$v = \frac{(b'Sb)^2}{\frac{1}{n_1^2}.\frac{(b'\Sigma_1 b)^2}{n_1 - 1} + \frac{1}{n_2^2}.\frac{(b'\Sigma_2 b)^2}{n_2 - 1}} \qquad \textbf{(18)}$$

Placing Equation (14) into Equation (18) to have:

$$v = \frac{\frac{(b'S_1 b)}{n_i} + \frac{(b'S_2 b)}{n_2}}{\frac{1}{n_1^2}.\frac{(b'S_1 b)^2}{n_1 - 1} + \frac{1}{n_2^2}.\frac{(b'S_2 b)^2}{n_2 - 1}} \qquad \textbf{(19)}$$

Set $b = S^{-1} y$ then Equation (19) becomes:

$$v = \frac{\left(\frac{y'S^{-1}S_1 S^{-1}y}{n_1} + \frac{y'S^{-1}S_2 S^{-1}y}{n_2} \right)^2}{\frac{(y'S^{-1}S_1 S^{-1}y)^2}{n_1^2(n_1 - 1)} + \frac{(y'S^{-1}S_2 S^{-1}y)^2}{n_2^2(n_2 - 1)}} \qquad \textbf{(20)}$$

Equation (20) can be in this form:

$$f = \frac{\left(\Sigma \frac{1}{n_i} (y S^{-1} S_i S^{-1} y) \right)^2}{\Sigma \frac{1}{n_i^2 (n_i - 1)} (y S^{-1} S_i S^{-1} y)^2} \qquad \textbf{(21)}$$

Let $y = \bar{X}_1 - \bar{X}_2$ then Equation (22) becomes:

$$f_{prop} = \frac{\left(\Sigma \frac{1}{n_i} \left((\bar{X}_1 - \bar{X}_2) S^{-1} S_i S^{-1} (\bar{X}_1 - \bar{X}_2) \right) \right)^2}{\Sigma \frac{1}{n_i^2 (n_i - 1)} \left((\bar{X}_1 - \bar{X}_2) S^{-1} S_i S^{-1} (\bar{X}_1 - \bar{X}_2) \right)^2}$$

$$T^2 \sim \left(\frac{f_{prop} \times p}{(f_{prop} - p + 1)} \right) F_{p, f_{prop} - p + 1} \qquad \textbf{(22)}$$

$$T_{prop} = \frac{(f_{prop} - p + 1) T^2}{f_{prop} \times p} \qquad \textbf{(23)}$$

Then Equation (23) is the test statistic of the proposed procedure. Statistical significance is then assessed by comparing the T_{Prop} statistic to its critical value $F_\alpha(p, f_{prop} - p + 1)$, that is, a critical value from the F distribution with p and $f_{prop} - p + 1$ degrees of freedom (df).

Simulation Study

A simulation study using R package was conducted in order to estimate and compare the Type I error rate and power for each of the previously discussed approximate solution (Johanson, Yao, Krishnamoorthy, Proposed procedure, Hotelling's T square, Nel and Van der Merwe and Yanagihara). The simulations are carried out when the null hypothesis is true and not true, for Multivariate normal distribution, when there are unequal variance – covariance matrix. Five (5) factors were varied in the simulation: the sample size, the number of variables p, variance co-variance matrices, mean vectors and significant levels. For each of the above combinations, an $n_i \times p$ data matrix X_i ($i = 1 \; and \; 2$) were replicated 1,000. The comparison criteria; type I error rate and power of the test were therefore obtained, and the results were presented in both tabular

The following are the levels used for each of the three factors.

Table 1: Levels used for Each of the Three Factors.

Multivariate Distribution	P	α	Sample size
Normal	2, 3, 4	0.01	20, 10
	2, 3, 4	0.025	50, 30
	2, 3, 4	0.05	100, 60

These levels provide 36 factor combinations the values for sample size are shown.

RESULTS

Table 2: Power of the Test.

P = 2
$x_1 = (20 \quad 30)$

$x_2 = (10 \quad 30)$

$S_1 = \begin{pmatrix} 267 & 200 \\ 200 & 267 \end{pmatrix}$

$S_2 = \begin{pmatrix} 35 & 25 \\ 25 & 35 \end{pmatrix}$

$\bar{x}_1 - \bar{x}_2 = (10 \quad 0)$

$n_1 \neq n_2$	Johan	Yao	Krish	Propo	Hotel	Nel	Yana
				α = 0.01			
20, 10	0.3852	0.3872	0.3872	0.3861	0.2146	**0.3915**	0.2487
50, 30	0.8332	0.8349	0.8346	**0.8374**	0.6672	0.8348	0.7691
100, 60	0.9910	0.9911	0.9911	**0.9912**	0.9497	0.9911	0.9865
				α = 0.025			
$n_1 \neq n_2$	Johan	Yao	Krish	Propo	Hotel	Nel	Yana
20, 10	0.5037	0.5043	0.5045	0.5037	0.3158	**0.5080**	0.3787
50, 30	0.8903	0.8911	0.8909	**0.8926**	0.7609	0.8910	0.8528
100, 60	0.9959	0.9959	0.9959	**0.9960**	0.9730	0.9959	0.9941
				α = 0.05			
$n_1 \neq n_2$	Johan	Yao	Krish	Propo	Hotel	Nel	Yana
20, 10	0.6106	0.6105	0.6104	0.6099	0.4201	**0.6133**	0.5092
50, 30	0.9319	0.9320	0.9319	**0.9329**	0.8348	0.9320	0.9110
100, 60	00.9982	0.9982	0.9982	**0.9983**	0.9855	0.9982	0.9975

From the Table 2, Nel and Van der Merwe has the highest power of the test when the sample sizes are small (20, 10) but at (50, 30) and (100, 60) the proposed procedure has the highest power than all other procedures.

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Table 3: Type I Error Rate.

P = 2
$x_1 = (20 \quad 30)$

$x_2 = (20 \quad 30)$

$S_1 = \begin{pmatrix} 267 & 200 \\ 200 & 267 \end{pmatrix}$

$S_2 = \begin{pmatrix} 35 & 25 \\ 25 & 35 \end{pmatrix}$

$\bar{x}_1 - \bar{x}_2 = (0 \quad 0)$

$n_1 \neq n_2$	Johan	Yao	Krish	Propo	Hotel	Nel	Yana
				α = 0.01			
20, 10	0.011	0.011	0.011	0.010	0.002	0.011	0.002
50, 30	0.007	0.007	0.007	0.007	0.001	0.006	0.004
100, 60	0.008	0.008	0.008	0.008	0.000	0.008	0.008
				α = 0.025			
$n_1 \neq n_2$	Johan	Yao	Krish	Propo	Hotel	Nel	Yana
20, 10	0.017	0.017	0.017	0.018	0.003	0.018	0.005
50, 30	0.026	0.027	0.026	0.027	0.008	0.026	0.015
100, 60	0.015	0.015	0.015	0.015	0.002	0.015	0.009
				α = 0.05			
$n_1 \neq n_2$	Johan	Yao	Krish	Propo	Hotel	Nel	Yana
20, 10	0.048	0.049	0.049	0.048	0.006	0.049	0.023
50, 30	0.045	0.044	0.045	0.046	0.011	0.045	0.035
100, 60	0.057	0.057	0.057	0.057	0.016	0.057	0.051

From Table 3, when sample size is (20, 10) the proposed procedure are on nominal level exactly while Hoteling T square and Yanagihara are below the nominal level, but at (50,30) and (100, 60) all the procedures are below the nominal level, at significant level 0.01. At α = 0.025, all the procedures are inflated at (50, 30) and deflated at (20, 10) and (100,60).

Table 4: Power of the Test.

P = 3

$x_1 = (30 \quad 24 \quad 50)$

$x_2 = (15 \quad 14 \quad 29)$

$S_1 = \begin{pmatrix} 500 & 450 & 350 \\ 450 & 500 & 180 \\ 350 & 180 & 500 \end{pmatrix}$

$S_2 = \begin{pmatrix} 90 & 30 & 20 \\ 30 & 90 & 10 \\ 20 & 10 & 90 \end{pmatrix}$

$\bar{x}_1 - \bar{x}_2 = (15 \quad 10 \quad 21)$

	Johan	Yao	Krish	Propo	Hotel	Nel	Yana
α = 0.01							
$n_1 \neq n_2$	Johan	Yao	Krish	Propo	Hotel	Nel	Yana
20, 10	0.2029	0.2198	0.2054	0.2147	0.1313	**0.2278**	0.0895
50, 30	0.6107	0.6182	0.6154	**0.6217**	0.4472	0.6203	0.5187
100, 60	0.9295	0.9312	0.9307	**0.9322**	0.8107	0.9315	0.9073
α = 0.025							
$n_1 \neq n_2$	Johan	Yao	Krish	Propo	Hotel	Nel	Yana
20, 10	0.2954	0.3120	0.2983	0.3062	0.2060	**0.3205**	0.1684
50, 30	0.7137	0.7195	0.7173	**0.7222**	0.5602	0.7209	0.6451
100, 60	0.9596	0.9606	0.9603	**0.9611**	0.8770	0.9608	0.9478
α = 0.05							
$n_1 \neq n_2$	Johan	Yao	Krish	Propo	Hotel	Nel	Yana
20, 10	0.4095	0.4255	0.4122	0.4207	0.3002	**0.4334**	0.2743
50, 30	0.7952	0.7992	0.7978	**0.8012**	0.6624	0.8003	0.7475
100, 60	0.9754	0.9760	0.9758	**0.9763**	0.9199	0.9761	0.9692

From Table 4, It is obvious that proposed procedure performed better than all other procedures at (50, 30) and (100, 60) but Nel and Van der Merwe is better when sample size is (20, 10) in all the scenarios considered.

Table 5: Type I Error Rate.

P = 3

$x_1 = (30 \quad 24 \quad 50)$

$x_2 = (15 \quad 14 \quad 29)$

$S_1 = \begin{pmatrix} 500 & 450 & 350 \\ 450 & 500 & 180 \\ 350 & 180 & 500 \end{pmatrix}$

$S_2 = \begin{pmatrix} 90 & 30 & 20 \\ 30 & 90 & 10 \\ 20 & 10 & 90 \end{pmatrix}$

$\bar{x}_1 - \bar{x}_2 = (15 \quad 10 \quad 21)$

	Johan	Yao	Krish	Propo	Hotel	Nel	Yana
α = 0.01							
$n_1 \neq n_2$	Johan	Yao	Krish	Propo	Hotel	Nel	Yana
20, 10	0.007	0.008	0.008	0.006	0.014	0.013	0.000
50, 30	0.011	0.010	0.011	0.090	0.013	0.011	0.006
100, 60	0.015	0.015	0.015	0.015	0.018	0.015	0.013
α = 0.025							
$n_1 \neq n_2$	Johan	Yao	Krish	Propo	Hotel	Nel	Yana
20, 10	0.035	0.033	0.035	0.027	0.033	0.041	0.003
50, 30	0.025	0.027	0.026	0.024	0.032	0.028	0.011
100, 60	0.029	0.031	0.031	0.030	0.039	0.031	0.026
α = 0.05							
$n_1 \neq n_2$	Johan	Yao	Krish	Propo	Hotel	Nel	Yana
20, 10	0.059	0.066	0.068	0.048	0.049	0.073	0.013
50, 30	0.053	0.059	0.055	0.056	0.052	0.057	0.042
100, 60	0.061	0.061	0.062	0.061	0.056	0.063	0.051

Table 5, the type I error rate of all procedures considered are fluctuating, either inflated or deflated. At significant level 0.01, 0.02, 0.05 there is inflation in type I error rate, when sample sizes are (50,30) and (100,60), but at (20, 10) all most all the procedures are deflated.

Table 6: Power of the Test.

P = 4

$$x_1 = (2.9 \quad 1.8 \quad 2.31 \quad 3.21)$$
$$x_2 = (1.89 \quad 0.8 \quad 0.3 \quad 0.2)$$
$$S_1 = \begin{pmatrix} 5 & 3 & 2 & 1 \\ 3 & 5 & 2 & 1 \\ 2 & 2 & 5 & 0.9 \\ 1 & 1 & 0.9 & 5 \end{pmatrix}$$
$$S_2 = \begin{pmatrix} 0.9 & 0.7 & 0.5 & 0.2 \\ 0.7 & 0.9 & 0.3 & 0.1 \\ 0.5 & 0.3 & 0.9 & 0.1 \\ 0.2 & 0.1 & 0.1 & 0.9 \end{pmatrix}$$
$$\bar{x}_1 - \bar{x}_2 = (1.01 \quad 1 \quad 2.10 \quad 3.01)$$

$n_1 \neq n_2$	Johan	Yao	Krish	Propo	Hotel	Nel	Yana
				α = 0.01			
20, 10	0.3430	0.3589	0.3599	0.3508	0.1974	**0.3737**	0.0780
50, 30	0.8600	0.8712	0.8711	**0.8750**	0.7027	0.8734	0.6825
100, 60	0.9968	0.9971	0.9971	**0.9973**	0.9733	0.9972	0.9902
				α = 0.025			
20, 10	0.4656	0.4800	0.4817	0.4721	0.2970	**0.4946**	0.1604
50, 30	0.9180	0.9249	0.9247	**0.9271**	0.7990	0.9260	0.8057
100, 60	0.9991	0.9992	0.9992	0.9992	0.9888	0.9992	0.9971
				α = 0.05			
20, 10	0.5662	0.5793	0.5804	0.5731	0.3905	**0.5914**	0.2585
50, 30	0.9507	0.9547	0.9547	**0.9561**	0.8649	0.9555	0.8860
100, 60	0.9995	0.9996	0.9996	0.9996	0.9938	0.9996	0.9987

Table 6 shows when that sample size is (20, 10) Nel and Van der Merwe performed better than other procedures, but when sample size increases to (50, 30) proposed procedure is better. And there was a great competition among the procedures at (100, 60).

Table 7: Type I Error Rate.

P = 4

$$x_1 = (2.9 \quad 1.8 \quad 2.31 \quad 3.21)$$
$$x_2 = (1.89 \quad 0.8 \quad 0.3 \quad 0.2)$$
$$S_1 = \begin{pmatrix} 5 & 3 & 2 & 1 \\ 3 & 5 & 2 & 1 \\ 2 & 2 & 5 & 0.9 \\ 1 & 1 & 0.9 & 5 \end{pmatrix}$$
$$S_2 = \begin{pmatrix} 0.9 & 0.7 & 0.5 & 0.2 \\ 0.7 & 0.9 & 0.3 & 0.1 \\ 0.5 & 0.3 & 0.9 & 0.1 \\ 0.2 & 0.1 & 0.1 & 0.9 \end{pmatrix}$$
$$\bar{x}_1 - \bar{x}_2 = (1.01 \quad 1 \quad 2.10 \quad 3.01)$$

$n_1 \neq n_2$	Johan	Yao	Krish	Propo	Hotel	Nel	Yana
				α = 0.01			
20, 10	0.008	0.013	0.011	0.011	0.002	0.013	0.000
50, 30	0.011	0.012	0.012	0.012	0.001	0.012	0.002
100, 60	0.007	0.007	0.007	0.007	0.000	0.007	0.004
				α = 0.025			
20, 10	0.024	0.029	0.031	0.026	0.004	0.033	0.000
50, 30	0.020	0.026	0.026	0.026	0.003	0.026	0.006
100, 60	0.021	0.022	0.022	0.022	0.002	0.022	0.010
				α = 0.05			
20, 10	0.050	0.055	0.055	0.051	0.009	0.060	0.000
50, 30	0.048	0.051	0.051	0.051	0.012	0.051	0.022
100, 60	0.052	0.053	0.052	0.053	0.008	0.053	0.034

Hotelling T square and Yanagihara are below the nominal level in all the scenarios considered, while other procedures fluctuated (Inflated or deflated) round the nominal level.

Data Set (For illustrated Example)

The data used here is an illustrated example used by Timm (1975). The two sample sizes considered are ten and twenty, respectively $(n_1 = 10 \text{ and } n_2 = 20)$ and two random variables ($p = 2$) form each population.

The sample means and their covariances are:

$$y_{1.} = \begin{pmatrix} 45.0 \\ 90.0 \end{pmatrix}, \qquad y_{2.} = \begin{pmatrix} 40.0 \\ 80.0 \end{pmatrix}$$

$$S_1 = \begin{pmatrix} 80.0 & 30.0 \\ 30.0 & 20.0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 120.0 & -100.0 \\ -100.0 & 200.0 \end{pmatrix}$$

$$n_1 = 10, \qquad n_2 = 20$$

The difference between the means is:

$$y = y_{1.} - y_{2.} = \begin{pmatrix} 5.0 \\ 10.0 \end{pmatrix}$$

And the test statistic is:

$$T^2 = (y_{1.} - y_{2.})' \left(\frac{S_1}{n_1} + \frac{S_2}{n_2} \right)^{-1} (y_{1.} - y_{2.})$$

$$(5.0, 10.0)' \begin{pmatrix} 0.0731707 & 0.0121951 \\ 0.0121951 & 0.0853659 \end{pmatrix} \begin{pmatrix} 5.0 \\ 10.0 \end{pmatrix}$$

$$T^2 = 11.58542$$

Table 8: Result from the Illustrated Example.

	Johan	Yao	Krish	Propo	Hotel	Nel	Yana
α = 0.05							
Critical.value	6.9780	7.2012	7.2230	7.7396	6.9567	6.9601	10.0088
Power	0.4979	0.5109	0.5121	**0.8680**	0.5068	0.4969	0.6244
α = 0.025							
Critical.value	8.8520	9.1661	9.1987	9.9867	8.7984	8.8036	13.8708
Power	0.6200	0.6334	0.6347	**0.9325**	0.6273	0.6180	0.7527
α = 0.01							
Critical.value	11.4986	11.9613	12.0129	13.2753	11.3828	11.3908	20.3456
Power	0.7503	0.7625	0.7638	**0.9732**	0.7551	0.7456	0.8680

From Table 8, the proposed procedure has the highest power followed by Yanagihara, Krishnamoorthy, Yao, Hotelling T square, Johanson, and Nel and Van der Merwe at all the significant level α considered (α = 0.05, 0.025 and 0.01).

FINDINGS

From the simulation, it is obvious from Table 2, 4 and 6 that when sample size are very small (20, 10) proposed procedure is not at his best, but when sample size increases to (50, 30) and (100, 60), the proposed procedure performed better than the all procedures considered. Nel and Van der Merwe performed better when sample size is very small (20, 10) followed by Yao, Krishmaorthy and Proposed procedures in term of power of the test in all the scenarios considered.

In terms of Type I error rate, proposed procedure competed favorably well with the other procedures selected for this study. Yao, Krishmaorthy, Johanson, Nel and Van der Merwe and the proposed procedures are fluctuating (inflated and deflated) around the nominal level while Hotellling T square and Yanagihara are below the nominal level.

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