On the Doubly Truncated Exponential Pareto Distribution

A.A. Akomolafe* and A. Maradesa

[1]Department of Statistics, Federal University of Technology Akure, PMB 704, Ondo State, Nigeria.
[2]The Ibarapa Polytechnic, Eruwa, Oyo State, Nigeria.

E-mail: akomolafeo1@yahoo.com*
maradprime1@gmail.com

ABSTRACT

This paper presents a mixture distribution of a new modelling tool which is termed the Doubly Truncated version of exponential Pareto distribution. The distribution allows us to capture some real characteristics of data and it is an important tool for understanding the phenomenon. The various statistical properties of this distribution were fully explored and discussed including the mean, variance, moments, mode, reliability function, and hazard function. The worth of the mixing distribution has been demonstrated by applying it to real life data.

(Keywords: doubly truncated exponential Pareto distribution, reliability function, hazard function, maximum likelihood estimation)

INTRODUCTION

Pareto distribution is widely used for modeling a wide range of data in several fields. It is very worthwhile for analyzing income distribution, actuarial, metrological data and equally preferable for survival analysis. Moreover, it is considered to be the most suitable choice as compared to other parameter distributions in several cases. Kareema, et al (2013) stated that:

$F(x;\alpha,\beta,\theta) = 1 - e^{-\alpha\left(\frac{x}{\beta}\right)^{\theta}}$ is the cdf of Exponential Pareto Distribution (EPD), then,

$f(x;\alpha,\beta,\theta) = \frac{\alpha\theta}{\beta}\left(\frac{x}{\beta}\right)^{\theta-1} e^{-\alpha\left(\frac{x}{\beta}\right)^{\theta}}$ is the pdf of EPD. from this, we can construct the doubly truncated version of exponential pareto by constraint the values of the random variable X to fall between the interval a$\leq x \leq b$. Where x is the random variable that follows the truncated distribution.

DERIVATION OF THE DOUBLY TRUNCATED EXPONENTIAL PARETO

Let g(x) = $f(x;\alpha,\beta,\theta)$ and G(x) = $F(x;\alpha,\beta,\theta)$, then $f_{DTEPD}\,(x;\tau) = \frac{\mathrm{g(x)}}{\mathrm{G(b)-G(a)}}$ **(1)**

By making necessary substitution, the Pdf of Doubly truncated exponential pareto distribution (DTEPD) will be obtained as (2).= $f_{DTEPD}(x;\alpha,\beta,\theta)$

$$= \frac{\frac{\alpha\theta}{\beta}\left(\frac{x}{\beta}\right)^{\theta-1} e^{-\alpha\left(\frac{x}{\beta}\right)^{\theta}}}{1-e^{-\alpha\left(\frac{b}{\beta}\right)^{\theta}} - 1 + e^{-\alpha\left(\frac{a}{\beta}\right)^{\theta}}} = \frac{\frac{\alpha\theta}{\beta}\left(\frac{x}{\beta}\right)^{\theta-1} e^{-\alpha\left(\frac{x}{\beta}\right)^{\theta}}}{e^{-\alpha\left(\frac{a}{\beta}\right)^{\theta}} - e^{-\alpha\left(\frac{b}{\beta}\right)^{\theta}}} \quad \text{for a} \leq x \leq b,\ \text{x>0},\ \alpha,\beta,\theta > 0 \qquad \textbf{(2)}$$

The (2) above is the probability density function of Doubly-truncated-exponential-pareto distribution. Also, its cdf can be obtained by making necessary substitution in $\frac{G(x)-G(a)}{G(b)-G(a)}$, this can be shown as follows:

$$=F_{DTEPD}(x;\alpha,\beta,\theta)=\frac{1-e^{-\alpha\left(\frac{x}{\beta}\right)^{\theta}}-1+e^{-\alpha\left(\frac{a}{\beta}\right)^{\theta}}}{1-e^{-\alpha\left(\frac{b}{\beta}\right)^{\theta}}-1+e^{-\alpha\left(\frac{a}{\beta}\right)^{\theta}}}=\frac{e^{-\alpha\left(\frac{a}{\beta}\right)^{\theta}}-e^{-\alpha\left(\frac{x}{\beta}\right)^{\theta}}}{e^{-\alpha\left(\frac{a}{\beta}\right)^{\theta}}-e^{-\alpha\left(\frac{b}{\beta}\right)^{\theta}}} \quad (3)$$

RELIABILITY

$$=R(t)=1-F_{DTEPD}(x;\alpha,\beta,\theta)=1-\left\{\frac{e^{-\alpha\left(\frac{a}{\beta}\right)^{\theta}}-e^{-\alpha\left(\frac{x}{\beta}\right)^{\theta}}}{e^{-\alpha\left(\frac{a}{\beta}\right)^{\theta}}-e^{-\alpha\left(\frac{b}{\beta}\right)^{\theta}}}\right\}$$

HAZARD FUNCTION

H(t)=

$$\frac{f_{DTEPD}(x;\alpha,\beta,\theta)}{R(t)}=\frac{\frac{\frac{\alpha\theta}{\beta}\left(\frac{x}{\beta}\right)^{\theta-1}e^{-\alpha\left(\frac{x}{\beta}\right)^{\theta}}}{e^{-\alpha\left(\frac{a}{\beta}\right)^{\theta}}-e^{-\alpha\left(\frac{b}{\beta}\right)^{\theta}}}}{1-\left\{\frac{e^{-\alpha\left(\frac{a}{\beta}\right)^{\theta}}-e^{-\alpha\left(\frac{x}{\beta}\right)^{\theta}}}{e^{-\alpha\left(\frac{a}{\beta}\right)^{\theta}}-e^{-\alpha\left(\frac{b}{\beta}\right)^{\theta}}}\right\}}=$$

$$\frac{\frac{\alpha\theta}{\beta}\left(\frac{x}{\beta}\right)^{\theta-1}e^{-\alpha\left(\frac{x}{\beta}\right)^{\theta}}}{e^{-\alpha\left(\frac{a}{\beta}\right)^{\theta}}-e^{-\alpha\left(\frac{b}{\beta}\right)^{\theta}}}\ x\ \frac{e^{-\alpha\left(\frac{a}{\beta}\right)^{\theta}}-e^{-\alpha\left(\frac{b}{\beta}\right)^{\theta}}}{e^{-\alpha\left(\frac{a}{\beta}\right)^{\theta}}-e^{-\alpha\left(\frac{b}{\beta}\right)^{\theta}}-\left\{\left(e^{-\alpha\left(\frac{a}{\beta}\right)^{\theta}}-e^{-\alpha\left(\frac{b}{\beta}\right)^{\theta}}\right)\left(e^{-\alpha\left(\frac{a}{\beta}\right)^{\theta}}-e^{-\alpha\left(\frac{x}{\beta}\right)^{\theta}}\right)\right\}} \quad (4)$$

$$\frac{\frac{\alpha\theta}{\beta}\left(\frac{x}{\beta}\right)^{\theta-1}e^{-\alpha\left(\frac{x}{\beta}\right)^{\theta}}}{e^{-\alpha\left(\frac{a}{\beta}\right)^{\theta}}-e^{-\alpha\left(\frac{b}{\beta}\right)^{\theta}}-\left\{\left(e^{-\alpha\left(\frac{a}{\beta}\right)^{\theta}}-e^{-\alpha\left(\frac{b}{\beta}\right)^{\theta}}\right)\left(e^{-\alpha\left(\frac{a}{\beta}\right)^{\theta}}-e^{-\alpha\left(\frac{x}{\beta}\right)^{\theta}}\right)\right\}}=$$

$$=\frac{1}{e^{-\alpha\left(\frac{a}{\beta}\right)^{\theta}}-e^{-\alpha\left(\frac{b}{\beta}\right)^{\theta}}}\left\{\frac{\frac{\alpha\theta}{\beta}\left(\frac{x}{\beta}\right)^{\theta-1}e^{-\alpha\left(\frac{x}{\beta}\right)^{\theta}}}{\left(1-\left(e^{-\alpha\left(\frac{a}{\beta}\right)^{\theta}}-e^{-\alpha\left(\frac{x}{\beta}\right)^{\theta}}\right)\right)}\right\} \quad (5)$$

ASYMPTOTIC PROPERTIES

Limit as $x\rightarrow\infty$

$$=\lim_{x\rightarrow\infty}\frac{\frac{\alpha\theta}{\beta}\left(\frac{x}{\beta}\right)^{\theta-1}e^{-\alpha\left(\frac{x}{\beta}\right)^{\theta}}}{e^{-\alpha\left(\frac{a}{\beta}\right)^{\theta}}-e^{-\alpha\left(\frac{b}{\beta}\right)^{\theta}}}=\lim_{x\rightarrow\infty}\frac{\frac{\alpha\theta}{\beta}(\infty)^{\theta-1}.\ 0}{e^{-\alpha\left(\frac{a}{\beta}\right)^{\theta}}-e^{-\alpha\left(\frac{b}{\beta}\right)^{\theta}}}=0$$

Limit as$x \to 0$

$$=\lim_{x\to 0}\frac{\frac{\alpha\theta}{\beta}\left(\frac{0}{\beta}\right)^{\theta-1}e^{-\alpha\left(\frac{x}{\beta}\right)^{\theta}}}{e^{-\alpha\left(\frac{a}{\beta}\right)^{\theta}}-e^{-\alpha\left(\frac{b}{\beta}\right)^{\theta}}}=\lim_{x\to 0}\frac{\frac{\alpha\theta}{\beta}(0)^{\theta-1}.\ 1}{e^{-\alpha\left(\frac{a}{\beta}\right)^{\theta}}-e^{-\alpha\left(\frac{b}{\beta}\right)^{\theta}}}=\frac{0}{e^{-\alpha\left(\frac{a}{\beta}\right)^{\theta}}-e^{-\alpha\left(\frac{b}{\beta}\right)^{\theta}}}=\mathbf{0}$$

There is an indication of at least one mode, since the limit of $f_{DTEPD}(x;\alpha,\beta,\theta)\ as\ x\to 0\ and\ x\to\infty\ are\ zero.$ Then DTEPD has a mode.

RENYI ENTROPY

Proposition 1

The Renyi entropy of DTEPD is given as:

$$\delta_r=\frac{1}{1-r}\log\left(e^{-\alpha\left(\frac{a}{\beta}\right)^{\theta}}-e^{-\alpha\left(\frac{b}{\beta}\right)^{\theta}}\right)^{-r}.\frac{(\alpha\theta)^r.\beta^{1-r}}{(\alpha r\theta)(\alpha r)^{r-\frac{r}{\theta}+\frac{1}{\theta}-1}}\Gamma\left(\frac{(\theta-1)(r-1)}{\theta}+1\right)$$

Proof

$$=\delta_r=\frac{1}{1-r}\log\int_0^\infty f^r(x)dx=\frac{1}{1-r}\log\int_0^\infty\left(\frac{\frac{\alpha\theta}{\beta}\left(\frac{0}{\beta}\right)^{\theta-1}e^{-\alpha\left(\frac{x}{\beta}\right)^{\theta}}}{e^{-\alpha\left(\frac{a}{\beta}\right)^{\theta}}-e^{-\alpha\left(\frac{b}{\beta}\right)^{\theta}}}\right)^r dx=$$

$$\frac{1}{1-r}\log(e^{-\alpha\left(\frac{a}{\beta}\right)^{\theta}}-e^{-\alpha\left(\frac{b}{\beta}\right)^{\theta}})^{-r}\int_0^\infty\left(\frac{\alpha\theta}{\beta}\right)^r\left(\left(\frac{x}{\beta}\right)^{\theta-1}\right)^r e^{-\alpha r\left(\frac{x}{\beta}\right)^{\theta}}dx \quad \textbf{(6)}$$

Let y = $\alpha r\left(\frac{x}{\beta}\right)^{\theta}$, $x=\beta\left(\frac{y}{\alpha r}\right)^{\frac{1}{\theta}}$ and $dx=\frac{dy}{\frac{\alpha r\theta}{\beta}\left(\frac{x}{\beta}\right)^{\theta-1}}$, therefore,

$$=\frac{1}{1-r}\log(e^{-\alpha\left(\frac{a}{\beta}\right)^{\theta}}-e^{-\alpha\left(\frac{b}{\beta}\right)^{\theta}})^{-r}\left(\frac{\alpha\theta}{\beta}\right)^r\int_0^\infty\left(\left(\frac{x}{\beta}\right)^{\theta-1}\right)^r.e^{-y}\frac{dy}{\frac{\alpha r\theta}{\beta}\left(\frac{x}{\beta}\right)^{\theta-1}} \quad \textbf{(7)}$$

$$=\frac{1}{1-r}\log(e^{-\alpha\left(\frac{a}{\beta}\right)^{\theta}}-e^{-\alpha\left(\frac{b}{\beta}\right)^{\theta}})^{-r}\left(\frac{\alpha\theta}{\beta}\right)^r\int_0^\infty\left(\left(\frac{x}{\beta}\right)^{\theta-1}\right)^r.e^{-y}\left(\frac{\alpha r\theta}{\beta}\right)^{-1}\left(\left(\frac{x}{\beta}\right)^{\theta-1}\right)^{-1}dy \quad \textbf{(8)}$$

$$=\frac{1}{1-r}\log(e^{-\alpha\left(\frac{a}{\beta}\right)^{\theta}}-e^{-\alpha\left(\frac{b}{\beta}\right)^{\theta}})^{-r}\left(\frac{\alpha\theta}{\beta}\right)^r.\left(\frac{\alpha r\theta}{\beta}\right)^{-1}\int_0^\infty\left(\left(\frac{x}{\beta}\right)^{\theta-1}\right)^{r-1}.e^{-y}\,dy \quad \textbf{(9)}$$

$$=\frac{1}{1-r}\log(e^{-\alpha\left(\frac{a}{\beta}\right)^{\theta}}-e^{-\alpha\left(\frac{b}{\beta}\right)^{\theta}})^{-r}\left(\frac{\alpha\theta}{\beta}\right)^r.\left(\frac{\alpha r\theta}{\beta}\right)^{-1}\int_0^\infty\frac{(x^{\theta-1})^{r-1}}{\beta^{(\theta-1)(r-1)}}e^{-y}\,dy \quad \textbf{(10)}$$

$$=\frac{1}{1-r}\log(e^{-\alpha\left(\frac{a}{\beta}\right)^{\theta}}-e^{-\alpha\left(\frac{b}{\beta}\right)^{\theta}})^{-r}\left(\frac{\alpha\theta}{\beta}\right)^{r}.\left(\frac{\alpha r\theta}{\beta}\right)^{-1}\beta^{-(r\theta-\theta-r+1)}\int_0^{\infty}(x^{\theta-1})^{r-1}\ e^{-y}\ dy \quad \textbf{(11)}$$

Put for $x=\beta\left(\frac{y}{\alpha r}\right)^{\frac{1}{\theta}}$ in (11)

$$=\frac{1}{1-r}\log(e^{-\alpha\left(\frac{a}{\beta}\right)^{\theta}}-e^{-\alpha\left(\frac{b}{\beta}\right)^{\theta}})^{-r}\left(\frac{\alpha\theta}{\beta}\right)^{r}.\left(\frac{\alpha r\theta}{\beta}\right)^{-1}\beta^{-(r\theta-\theta-r+1)}\int_0^{\infty}\left(\left(\beta\left(\frac{y}{\alpha r}\right)^{\frac{1}{\theta}}\right)^{\theta-1}\right)^{r-1}\ e^{-y}\ dy \quad \textbf{(12)}$$

$$=\frac{1}{1-r}\log(e^{-\alpha\left(\frac{a}{\beta}\right)^{\theta}}-e^{-\alpha\left(\frac{b}{\beta}\right)^{\theta}})^{-r}\left(\frac{\alpha\theta}{\beta}\right)^{r}.\left(\frac{\alpha r\theta}{\beta}\right)^{-1}\beta^{-(r\theta-\theta-r+1)}\int_0^{\infty}\beta^{(r-1)(\theta-1)}\frac{y^{\left(1-\frac{1}{\theta}\right)(r-1)}}{(\alpha r)^{\left(1-\frac{1}{\theta}\right)(r-1)}}\ e^{-y}\ dy \quad \textbf{(13)}$$

$$=\frac{1}{1-r}\log(e^{-\alpha\left(\frac{a}{\beta}\right)^{\theta}}-e^{-\alpha\left(\frac{b}{\beta}\right)^{\theta}})^{-r}\left(\frac{\alpha\theta}{\beta}\right)^{r}.\left(\frac{\alpha r\theta}{\beta}\right)^{-1}\beta^{-(r\theta-\theta-r+1)}\frac{\beta^{(r-1)(\theta-1)}}{(\alpha r)^{r-1+\frac{r}{\theta}-\frac{1}{\theta}}}\int_0^{\infty}y^{\left(1-\frac{1}{\theta}\right)(r-1)}\ e^{-y}\ dy \quad \textbf{(14)}$$

$$=\frac{1}{1-r}\log(e^{-\alpha\left(\frac{a}{\beta}\right)^{\theta}}-e^{-\alpha\left(\frac{b}{\beta}\right)^{\theta}})^{-r}\left(\frac{\alpha\theta}{\beta}\right)^{r}.\left(\frac{\alpha r\theta}{\beta}\right)^{-1}\beta^{-(r\theta-\theta-r+1)}\frac{\beta^{(r-1)(\theta-1)}}{(\alpha r)^{r-1-\frac{r}{\theta}+\frac{1}{\theta}}}\int_0^{\infty}y^{r-\frac{r}{\theta}+\frac{1}{\theta}-1}\ e^{-y}\ dy \quad \textbf{(15)}$$

$$=\frac{1}{1-r}log(e^{-\alpha\left(\frac{a}{\beta}\right)^{\theta}}-e^{-\alpha\left(\frac{b}{\beta}\right)^{\theta}})^{-r}\left(\frac{\alpha\theta}{\beta}\right)^{r}.\left(\frac{\alpha r\theta}{\beta}\right)^{-1}\beta^{-(r\theta-\theta-r+1)}\frac{\beta^{r\theta-r-\theta+1}}{(\alpha r)^{r-1-\frac{r}{\theta}+\frac{1}{\theta}}}\Gamma\left(\frac{(\theta-1)(r-1)}{\theta}+1\right) \quad \textbf{(16)}$$

$$=\frac{1}{1-r}log(e^{-\alpha\left(\frac{a}{\beta}\right)^{\theta}}-e^{-\alpha\left(\frac{b}{\beta}\right)^{\theta}})^{-r}\left(\frac{\alpha\theta}{\beta}\right)^{r}\left(\frac{\alpha r\theta}{\beta}\right)^{-1}\beta^{(-r\theta+\theta+r-1)}\frac{\beta^{r\theta-r-\theta+1}}{(\alpha r\theta)(\alpha r)^{r-1-\frac{r}{\theta}+\frac{1}{\theta}}}\ \Gamma\left(\frac{(\theta-1)(r-1)}{\theta}+1\right) \quad \textbf{(17)}$$

$$=\frac{1}{1-r}log(e^{-\alpha\left(\frac{a}{\beta}\right)^{\theta}}-e^{-\alpha\left(\frac{b}{\beta}\right)^{\theta}})^{-r}\left(\frac{\alpha\theta}{\beta}\right)^{r}\left(\frac{\alpha r\theta}{\beta}\right)^{-1}\frac{1}{(\alpha r\theta)(\alpha r)^{r-1-\frac{r}{\theta}+\frac{1}{\theta}}}\Gamma\left(\frac{(\theta-1)(r-1)}{\theta}+1\right) \quad \textbf{(18)}$$

$$=\frac{1}{1-r}log(e^{-\alpha\left(\frac{a}{\beta}\right)^{\theta}}-e^{-\alpha\left(\frac{b}{\beta}\right)^{\theta}})^{-r}\left(\frac{\alpha\theta}{\beta}\right)^{r}\frac{\beta}{(\alpha r\theta)(\alpha r)^{r-\frac{r}{\theta}+\frac{1}{\theta}-1}}\Gamma\left(\frac{(\theta-1)(r-1)}{\theta}+1\right) \quad \textbf{(19)}$$

$$=\frac{1}{1-r}\log\left(e^{-\alpha\left(\frac{a}{\beta}\right)^{\theta}}-e^{-\alpha\left(\frac{b}{\beta}\right)^{\theta}}\right)^{-r}(\alpha\theta)^{r}.\frac{\beta^{-r}\beta}{(\alpha r\theta)(\alpha r)^{r-\frac{r}{\theta}+\frac{1}{\theta}-1}}\Gamma\left(\frac{(\theta-1)(r-1)}{\theta}+1\right) \quad \textbf{(20)}$$

$$\boldsymbol{\delta_r}=\frac{1}{1-r}\log\left(e^{-\alpha\left(\frac{a}{\beta}\right)^{\theta}}-e^{-\alpha\left(\frac{b}{\beta}\right)^{\theta}}\right)^{-r}.\frac{(\alpha\theta)^{r}.\beta^{1-r}}{(\alpha r\theta)(\alpha r)^{r-\frac{r}{\theta}+\frac{1}{\theta}-1}}\Gamma\left(\frac{(\theta-1)(r-1)}{\theta}+1\right) \quad \textbf{(21)}$$

Equation (21) above represent the Renyi entropy of DTEPD.

$\beta - Entropy$

The Renyi entropy of DTEPD is given as:

$$\delta_{\bar{\beta}} = \frac{1}{\bar{\beta}-1}\left(e^{-\alpha\left(\frac{a}{\beta}\right)^{\theta}} - e^{-\alpha\left(\frac{b}{\beta}\right)^{\theta}}\right)^{-\bar{\beta}} . \frac{\beta^{1-\bar{\beta}}(\alpha\theta)^{\bar{\beta}}}{(\alpha\bar{\beta}\theta)(\alpha\bar{\beta})^{\bar{\beta}-\frac{\bar{\beta}}{\theta}+\frac{1}{\theta}-1}} \Gamma\left(\frac{(\theta-1)(\bar{\beta}-1)}{\theta} + 1\right)$$

$$=\delta_{\bar{\beta}} = \frac{1}{\bar{\beta}-1}\int_0^{\infty} f^{\bar{\beta}}\, dx \;=\frac{1}{\bar{\beta}-1}\int_0^{\infty}\left(\frac{\frac{\alpha\theta}{\beta}\left(\frac{x}{\beta}\right)^{\theta-1} e^{-\alpha\left(\frac{x}{\beta}\right)^{\theta}}}{e^{-\alpha\left(\frac{a}{\beta}\right)^{\theta}} - e^{-\alpha\left(\frac{b}{\beta}\right)^{\theta}}}\right)^{\bar{\beta}} dx \quad \textbf{(22)}$$

$$= \frac{1}{\bar{\beta}-1}(e^{-\alpha\left(\frac{a}{\beta}\right)^{\theta}} - e^{-\alpha\left(\frac{b}{\beta}\right)^{\theta}})^{-\bar{\beta}} \int_0^{\infty}\left(\frac{\alpha\theta}{\beta}\right)^{\bar{\beta}}\left(\left(\frac{x}{\beta}\right)^{\theta-1}\right)^{\bar{\beta}} e^{-\alpha\bar{\beta}\left(\frac{x}{\beta}\right)^{\theta}} dx \quad \textbf{(23)}$$

= Let y = $\alpha r\left(\frac{x}{\beta}\right)^{\theta}$ $, x = \beta\left(\frac{y}{\alpha\bar{\beta}}\right)^{\frac{1}{\theta}}$ and $dx = \frac{dy}{\frac{\alpha\bar{\beta}\theta}{\beta}\left(\frac{x}{\beta}\right)^{\theta-1}}$, therefore,

$$=\frac{1}{\bar{\beta}-1}(e^{-\alpha\left(\frac{a}{\beta}\right)^{\theta}} - e^{-\alpha\left(\frac{b}{\beta}\right)^{\theta}})^{-\bar{\beta}}\left(\frac{\alpha\theta}{\beta}\right)^{\bar{\beta}}\int_0^{\infty}\left(\left(\frac{x}{\beta}\right)^{\theta-1}\right)^{\bar{\beta}} . e^{-y}\,\frac{dy}{\frac{\alpha\bar{\beta}\theta}{\beta}\left(\frac{x}{\beta}\right)^{\theta-1}} \quad \textbf{(24)}$$

$$=\frac{1}{\bar{\beta}-1}(e^{-\alpha\left(\frac{a}{\beta}\right)^{\theta}} - e^{-\alpha\left(\frac{b}{\beta}\right)^{\theta}})^{-\bar{\beta}}\left(\frac{\alpha\theta}{\beta}\right)^{\bar{\beta}}\int_0^{\infty}\left(\left(\frac{x}{\beta}\right)^{\theta-1}\right)^{\bar{\beta}} . e^{-y}\left(\frac{\alpha\bar{\beta}\theta}{\beta}\right)^{-1}\left(\left(\frac{x}{\beta}\right)^{\theta-1}\right)^{-1} dy \quad \textbf{(25)}$$

$$=\frac{1}{\bar{\beta}-1}(e^{-\alpha\left(\frac{a}{\beta}\right)^{\theta}} - e^{-\alpha\left(\frac{b}{\beta}\right)^{\theta}})^{-\bar{\beta}}\left(\frac{\alpha\theta}{\beta}\right)^{\bar{\beta}} . \left(\frac{\alpha\bar{\beta}\theta}{\beta}\right)^{-1}\int_0^{\infty}\left(\left(\frac{x}{\beta}\right)^{\theta-1}\right)^{\bar{\beta}-1} . e^{-y}\, dy \quad \textbf{(26)}$$

$$=\frac{1}{\bar{\beta}-1}(e^{-\alpha\left(\frac{a}{\beta}\right)^{\theta}} - e^{-\alpha\left(\frac{b}{\beta}\right)^{\theta}})^{-\bar{\beta}}\left(\frac{\alpha\theta}{\beta}\right)^{r} . \left(\frac{\alpha r\theta}{\beta}\right)^{-1}\int_0^{\infty}\frac{(x^{\theta-1})^{\bar{\beta}-1}}{\beta^{(\theta-1)(\bar{\beta}-1)}} e^{-y}\, dy \quad \textbf{(27)}$$

$$=\frac{1}{\bar{\beta}-1}(e^{-\alpha\left(\frac{a}{\beta}\right)^{\theta}} - e^{-\alpha\left(\frac{b}{\beta}\right)^{\theta}})^{-\bar{\beta}}\left(\frac{\alpha\theta}{\beta}\right)^{\bar{\beta}} . \left(\frac{\alpha\bar{\beta}\theta}{\beta}\right)^{-1}\beta^{-(\bar{\beta}\theta-\theta-\bar{\beta}+1)}\int_0^{\infty}(x^{\theta-1})^{\bar{\beta}-1} e^{-y}\, dy \quad \textbf{(28)}$$

Put for $x = \beta\left(\frac{y}{\alpha\bar{\beta}}\right)^{\frac{1}{\theta}}$ in **(29)**

$$=\frac{1}{\bar{\beta}-1}(e^{-\alpha\left(\frac{a}{\beta}\right)^{\theta}} - e^{-\alpha\left(\frac{b}{\beta}\right)^{\theta}})^{-\bar{\beta}}\left(\frac{\alpha\theta}{\beta}\right)^{\bar{\beta}} . \left(\frac{\alpha\bar{\beta}\theta}{\beta}\right)^{-1}\beta^{-(\theta\bar{\beta}-\theta-\bar{\beta}+1)}\int_0^{\infty}\left(\left(\beta\left(\frac{y}{\alpha\bar{\beta}}\right)^{\frac{1}{\theta}}\right)^{\theta-1}\right)^{\bar{\beta}-1} e^{-y}\, dy \quad \textbf{(30)}$$

$$=\frac{1}{\bar{\beta}-1}(e^{-\alpha\left(\frac{a}{\beta}\right)^{\theta}} - e^{-\alpha\left(\frac{b}{\beta}\right)^{\theta}})^{-\bar{\beta}}\left(\frac{\alpha\theta}{\beta}\right)^{\bar{\beta}} . \left(\frac{\alpha\bar{\beta}\theta}{\beta}\right)^{-1}\beta^{-(\theta\bar{\beta}-\theta-\bar{\beta}+1)}\int_0^{\infty}\beta^{\overline{(\beta}-1)(\theta-1)}\frac{y^{\left(1-\frac{1}{\theta}\right)(\bar{\beta}-1)}}{(\alpha\bar{\beta})^{\left(1-\frac{1}{\theta}\right)(\bar{\beta}-1)}} e^{-y}\, dy \quad \textbf{(31)}$$

$$=\frac{1}{\bar{\beta}-1}(e^{-\alpha\left(\frac{a}{\beta}\right)^{\theta}}-e^{-\alpha\left(\frac{b}{\beta}\right)^{\theta}})^{-\bar{\beta}}\left(\frac{\alpha\theta}{\beta}\right)^{\bar{\beta}}.\left(\frac{\alpha\bar{\beta}\theta}{\beta}\right)^{-1}\beta^{-(\theta\bar{\beta}-\theta-\bar{\beta}+1)}\frac{\beta^{\theta\bar{\beta}-\bar{\beta}-\theta+1}}{(\alpha r)^{\bar{\beta}-1+\frac{\bar{\beta}}{\theta}-\frac{1}{\theta}}}\int_0^\infty y^{\left(1-\frac{1}{\theta}\right)(\bar{\beta}-1)}e^{-y}\,dy \quad \textbf{(32)}$$

$$=\frac{1}{\bar{\beta}-1}(e^{-\alpha\left(\frac{a}{\beta}\right)^{\theta}}-e^{-\alpha\left(\frac{b}{\beta}\right)^{\theta}})^{-\bar{\beta}}\left(\frac{\alpha\theta}{\beta}\right)^{\bar{\beta}}.\left(\frac{\alpha\bar{\beta}\theta}{\beta}\right)^{-1}\beta^{-(\theta\bar{\beta}-\theta-\bar{\beta}+1)}\frac{\beta^{\theta\bar{\beta}-\bar{\beta}-\theta+1}}{(\alpha r)^{\bar{\beta}-1-\frac{\bar{\beta}}{\theta}+\frac{1}{\theta}}}\int_0^\infty y^{\bar{\beta}-\frac{\bar{\beta}}{\theta}+\frac{1}{\theta}-1}e^{-y}\,dy \quad \textbf{(33)}$$

$$=\frac{1}{\bar{\beta}-1}(e^{-\alpha\left(\frac{a}{\beta}\right)^{\theta}}-e^{-\alpha\left(\frac{b}{\beta}\right)^{\theta}})^{-\bar{\beta}}\left(\frac{\alpha\theta}{\beta}\right)^{\bar{\beta}}.\left(\frac{\alpha\bar{\beta}\theta}{\beta}\right)^{-1}\beta^{-\theta\bar{\beta}+\theta+\bar{\beta}-1)}\frac{\beta^{\theta\bar{\beta}-\bar{\beta}-\theta+1}}{(\alpha r)^{\bar{\beta}-1-\frac{\bar{\beta}}{\theta}+\frac{1}{\theta}}}\Gamma\left(\frac{(\theta-1)(\bar{\beta}-1)}{\theta}+1\right) \quad \textbf{(34)}$$

$$=\frac{1}{\bar{\beta}-1}(e^{-\alpha\left(\frac{a}{\beta}\right)^{\theta}}-e^{-\alpha\left(\frac{b}{\beta}\right)^{\theta}})^{-\bar{\beta}}\left(\frac{\alpha\theta}{\beta}\right)^{\bar{\beta}}\frac{\beta}{(\alpha\bar{\beta}\theta)(\alpha\bar{\beta})^{\bar{\beta}-1-\frac{\bar{\beta}}{\theta}+\frac{1}{\theta}}}\Gamma\left(\frac{(\theta-1)(\bar{\beta}-1)}{\theta}+1\right) \quad \textbf{(35)}$$

$$=\frac{1}{\bar{\beta}-1}(e^{-\alpha\left(\frac{a}{\beta}\right)^{\theta}}-e^{-\alpha\left(\frac{b}{\beta}\right)^{\theta}})^{-\bar{\beta}}\left(\frac{\alpha\theta}{\beta}\right)^{\bar{\beta}}\frac{\beta}{(\alpha\bar{\beta}\theta)(\alpha\bar{\beta})^{\bar{\beta}-1-\frac{\bar{\beta}}{\theta}+\frac{1}{\theta}}}\Gamma\left(\frac{(\theta-1)(\bar{\beta}-1)}{\theta}+1\right) \quad \textbf{(36)}$$

$$=\frac{1}{\bar{\beta}-1}\left(e^{-\alpha\left(\frac{a}{\beta}\right)^{\theta}}-e^{-\alpha\left(\frac{b}{\beta}\right)^{\theta}}\right)^{-\bar{\beta}}.\frac{\beta\beta^{-\bar{\beta}}(\alpha\theta)^{\bar{\beta}}}{(\alpha\bar{\beta}\theta)(\alpha\bar{\beta})^{\bar{\beta}-\frac{\bar{\beta}}{\theta}+\frac{1}{\theta}-1}}\Gamma\left(\frac{(\theta-1)(\bar{\beta}-1)}{\theta}+1\right) \quad \textbf{(37)}$$

$$\boldsymbol{\delta_{\bar{\beta}}}=\frac{1}{\bar{\beta}-1}\left(e^{-\alpha\left(\frac{a}{\beta}\right)^{\theta}}-e^{-\alpha\left(\frac{b}{\beta}\right)^{\theta}}\right)^{-\bar{\beta}}.\frac{\beta^{1-\bar{\beta}}(\alpha\theta)^{\bar{\beta}}}{(\alpha\bar{\beta}\theta)(\alpha\bar{\beta})^{\bar{\beta}-\frac{\bar{\beta}}{\theta}+\frac{1}{\theta}-1}}\Gamma\left(\frac{(\theta-1)(\bar{\beta}-1)}{\theta}+1\right) \quad \textbf{(38)}$$

The Equation (38) above is the $\beta - entropy$ of DTEPD.

MOMENT

$$=\boldsymbol{E(x^r)}=\int_0^\infty \boldsymbol{x^r f(x)dx} = \int_0^\infty \boldsymbol{x^r}\frac{\frac{\alpha\theta}{\beta}\left(\frac{x}{\beta}\right)^{\theta-1}e^{-\alpha\left(\frac{x}{\beta}\right)^{\theta}}}{e^{-\alpha\left(\frac{a}{\beta}\right)^{\theta}}-e^{-\alpha\left(\frac{b}{\beta}\right)^{\theta}}}\,dx \quad \textbf{(39)}$$

$$=\left(e^{-\alpha\left(\frac{a}{\beta}\right)^{\theta}}-e^{-\alpha\left(\frac{b}{\beta}\right)^{\theta}}\right)^{-1}\int_0^\infty \boldsymbol{x^r}\frac{\alpha\theta}{\beta}\left(\frac{x}{\beta}\right)^{\theta-1}e^{-\alpha\left(\frac{x}{\beta}\right)^{\theta}}dx \quad \textbf{(40)}$$

$$=\left(e^{-\alpha\left(\frac{a}{\beta}\right)^{\theta}}-e^{-\alpha\left(\frac{b}{\beta}\right)^{\theta}}\right)^{-1}\frac{\alpha\theta}{\beta}\int_0^\infty \boldsymbol{x^r}\left(\frac{x}{\beta}\right)^{\theta-1}e^{-\alpha\left(\frac{x}{\beta}\right)^{\theta}}dx \quad \textbf{(41)}$$

Substitute for y = $\alpha\left(\frac{x}{\beta}\right)^{\theta}$; $x = \beta\left(\frac{y}{\alpha}\right)^{\frac{1}{\theta}}$ and $dx = \frac{dy}{\frac{\alpha\theta}{\beta}\left(\frac{x}{\beta}\right)^{\theta-1}}$ in

$$\left(e^{-\alpha\left(\frac{a}{\beta}\right)^{\theta}} - e^{-\alpha\left(\frac{b}{\beta}\right)^{\theta}}\right)^{-1} \frac{\alpha\theta}{\beta}\int_0^{\infty} x^r \left(\frac{x}{\beta}\right)^{\theta-1} . e^{-y} \frac{dy}{\frac{\alpha\theta}{\beta}\left(\frac{x}{\beta}\right)^{\theta-1}} = \left(e^{-\alpha\left(\frac{a}{\beta}\right)^{\theta}} - e^{-\alpha\left(\frac{b}{\beta}\right)^{\theta}}\right)^{-1} \int_0^{\infty} x^r e^{-y}\, dy \quad (42)$$

$$= \left(e^{-\alpha\left(\frac{a}{\beta}\right)^{\theta}} - e^{-\alpha\left(\frac{b}{\beta}\right)^{\theta}}\right)^{-1} \int_0^{\infty} \left(\beta \left(\frac{y}{\alpha}\right)^{\frac{1}{\theta}}\right)^r e^{-y}\, dy \quad (43)$$

$$= \left(e^{-\alpha\left(\frac{a}{\beta}\right)^{\theta}} - e^{-\alpha\left(\frac{b}{\beta}\right)^{\theta}}\right)^{-1} \frac{\beta^r}{(\alpha)^{\frac{r}{\theta}}} \int_0^{\infty} y^{\frac{r}{\theta}} e^{-y}\, dy = \left(e^{-\alpha\left(\frac{a}{\beta}\right)^{\theta}} - e^{-\alpha\left(\frac{b}{\beta}\right)^{\theta}}\right)^{-1} \frac{\beta^r}{(\alpha)^{\frac{r}{\theta}}} \Gamma\left(\frac{r}{\theta}+1\right) \quad (44)$$

$$\boldsymbol{E(x^r)} = \left(e^{-\alpha\left(\frac{a}{\beta}\right)^{\theta}} - e^{-\alpha\left(\frac{b}{\beta}\right)^{\theta}}\right)^{-1} \left(\frac{\beta}{\sqrt[\theta]{\alpha}}\right)^r \Gamma\left(\frac{r}{\theta}+1\right) \quad ,r = 1,2,3\ldots \quad (45)$$

$$=\boldsymbol{E(x)} = \left(e^{-\alpha\left(\frac{a}{\beta}\right)^{\theta}} - e^{-\alpha\left(\frac{b}{\beta}\right)^{\theta}}\right)^{-1} \frac{\beta}{\sqrt[\theta]{\alpha}} \Gamma\left(\frac{1}{\theta}+1\right) \quad (46)$$

$$= \boldsymbol{E(x^2)} = \left(e^{-\alpha\left(\frac{a}{\beta}\right)^{\theta}} - e^{-\alpha\left(\frac{b}{\beta}\right)^{\theta}}\right)^{-1} \left(\frac{\beta}{\sqrt[\theta]{\alpha}}\right)^2 \Gamma\left(\frac{2}{\theta}+1\right) \quad (47)$$

$$\boldsymbol{E(x^3)} = \left(e^{-\alpha\left(\frac{a}{\beta}\right)^{\theta}} - e^{-\alpha\left(\frac{b}{\beta}\right)^{\theta}}\right)^{-1} \left(\frac{\beta}{\sqrt[\theta]{\alpha}}\right)^3 \Gamma\left(\frac{3}{\theta}+1\right) \quad (48)$$

$$=\boldsymbol{E(x^4)} = \left(e^{-\alpha\left(\frac{a}{\beta}\right)^{\theta}} - e^{-\alpha\left(\frac{b}{\beta}\right)^{\theta}}\right)^{-1} \left(\frac{\beta}{\sqrt[\theta]{\alpha}}\right)^4 \Gamma\left(\frac{4}{\theta}+1\right) \quad (49)$$

Var(x) = $\boldsymbol{E(x^2)} - \left(\boldsymbol{E(x)}\right)^2$ =

$$\left(e^{-\alpha\left(\frac{a}{\beta}\right)^{\theta}} - e^{-\alpha\left(\frac{b}{\beta}\right)^{\theta}}\right)^{-1} \left(\frac{\beta}{\sqrt[\theta]{\alpha}}\right)^2 \Gamma\left(\frac{2}{\theta}+1\right) - \left(\left(e^{-\alpha\left(\frac{a}{\beta}\right)^{\theta}} - e^{-\alpha\left(\frac{b}{\beta}\right)^{\theta}}\right)^{-1} \frac{\beta}{\sqrt[\theta]{\alpha}} \Gamma\left(\frac{1}{\theta}+1\right)\right)^2 \quad (50)$$

MAXIMUM LIKELIHOOD ESTIMATION

$$= \boldsymbol{Lf(x;\tau)} = \prod_{i=1}^{n} \frac{\frac{\alpha\theta}{\beta}\left(\frac{x}{\beta}\right)^{\theta-1} e^{-\alpha\left(\frac{x}{\beta}\right)^{\theta}}}{e^{-\alpha\left(\frac{a}{\beta}\right)^{\theta}} - e^{-\alpha\left(\frac{b}{\beta}\right)^{\theta}}} = \prod_{i=1}^{n} \left(e^{-\alpha\left(\frac{a}{\beta}\right)^{\theta}} - e^{-\alpha\left(\frac{b}{\beta}\right)^{\theta}}\right)^{-1} \frac{\alpha\theta}{\beta}\left(\frac{x}{\beta}\right)^{\theta-1} e^{-\alpha\left(\frac{x}{\beta}\right)^{\theta}} \quad (51)$$

$$= \boldsymbol{Lf(x;\tau)} = \left(e^{-\alpha\left(\frac{a}{\beta}\right)^{\theta}} - e^{-\alpha\left(\frac{b}{\beta}\right)^{\theta}}\right)^{-n} \alpha^n \theta^n \beta^{-n} \left(\frac{\prod_{i=1}^{n} x_i}{\beta^n}\right)^{\theta-1} e^{-\alpha\frac{\sum_{i=1}^{n} x_i{}^{\theta}}{\beta^{\theta}}} \quad (52)$$

=ln

$$\boldsymbol{Lf}(\boldsymbol{x};\boldsymbol{\tau}) = -nln\left(e^{-\alpha\left(\frac{a}{\beta}\right)^{\theta}} - e^{-\alpha\left(\frac{b}{\beta}\right)^{\theta}}\right) + nln\alpha + nln\theta - nln\beta + (\theta-1)ln\prod_{i=1}^{n} x_i - n(\theta-1)ln\beta - -\alpha\frac{\sum_{i=1}^{n} x_i{}^{\theta}}{\beta^{\theta}}$$

(53)

From (53), the parameters of DTEPD can be estimated via numerical method (Newton Raphson algorithm) and the information matrix can be used to test for the stability of the parameters and thus contributing to the reliability of the model over its parent distribution in modeling pattern or behavior of certain random process or stochastic phenomenon at a specified interval (a$\le x \le b$).

CONCLUSIONS

By compounding two or more probability distributions, we get the corresponding hybrid distribution with increased number of parameters which is believed to give the newly compounded distribution more flexibility, consistency, stability, sufficiency uniqueness and wider applicability as compare to its parent distribution. Therefore, the hybrid Doubly Truncated exponential Pareto distribution is said to have vast applicability extending beyond modeling simple Brownian movement but can be used to model statistical behavior of stochastic processes such as studying the consumers' buying behavior and in complex epidemiological studies because of its increased number of parameters which give the hybrid distribution more flexibility to model many stochastic phenomena.

REFERENCES

1. Alzaatreh, A., C. Lee, and F. Famoye. 2013. "A New Method for Generating Families of Continuous Distributions". *Metron*. 71(1): 63–79.

2. Akinsete, A., F. Famoye, and C. Lee. 2008. "The Beta-Pareto Distribution". *Statistics*. 42(6): 547-563.

3. Akomolafe, A.A. and A. Maradesa. 2017. "Beta-Half-normal Distribution and Its Properties". *International Journal of Advance Research and Publication*. 4:17-22.

4. Ashour, S.K. and M.A. Eltehiwy. 2013. "Transmuted Lomax Distribution". *America Journal of Applied Mathematics and Statistics*. 1(6):121-127.

5. Badmus, N.I., M. Ikegwu, and O. Emmanuel. 2013. "The Beta-Weighted Weibull Distribution: Some Properties and Application to Bladder Cancer Data". *Applied and Computational Mathematics*. 2:5.

6. Barreto-Souza, W., G.M. Cordeiro and A.B. Simas. 2011. "Some results for Beta Frechet Distribution". *Communications in Statistics Theory & Methods*. 40:798–811.

7. Blackwell, R.D., R.D. Minar, and P.W. Engel. 2001. *Consumer Behavior*. Harcourt College Publishers: New York, NY.

8. Brown, M. 2003. "Buying or Browsing? An Exploration of Shopping Orientations and Online Purchase Intention". *European Journal of Marketing*. 37(11/12):1666-1684.

9. Burr, I.W. 1942. "Cumulative Frequency Functions". *Annals of Mathematical Statistics*. 13: 215-232.

10. Cordeiro, G.M., and M. de Castro. 2011. "A New Family of Generalized Distributions". *Journal of Statistical Computation and Simulation*. 81(7): 883-898.

11. Cordeiro, G.M., A.B. Simas, and B.D. Stosic. 2011. "Closed Form Expressions for Moments of the Beta Weibull Distribution". *Annals of the Brazilian Academy of Sciences*. 8(83):357-373.

12. Cordeiro, G.M., C. Alexandra, J.M. Ortega, M.M. Edwin, and J.M. Sarabia. 2011. "Generated Distribution". ICMA Centre. Discussion Paper in Finance. 1-29.

13. Dantas, M.I., V.P. Minim, R. Deliza, and R. Pusehman. 2004. "The Effect of Packaging on the Perception of Minimally Processed Products". *Journal of International Food and Agribusiness Marketing*. 16(2): 71-83.

14. D'sounza, C., M. Taghian, and M. Khosla. 2007. "Examination of Environmental Beliefs and its Impact on the Influence on the Performance of Price, Quality and Demographic Characteristics with Respect to Green Purchase Intention".

Journal of Targeting, Measurement and Analysis for Marketing, 15(2):69-78.

15. Eugene, N., C. Lee, and F. Famoye. 2002. “Beta-Normal Distribution and its Applications”. *Communications in Statistic: Theory and Methods*. 31(4): 497-512.

16. Famoye, F., C. Lee, and O. Olumolade. 2005. “The Beta-Weibull Distribution”. *Journal of Statistical Theory*. 4:121–136.

17. Famoye, F., C. Lee, and N. Eugene. 2004. “Betanormal Distribution: Bimodality Properties and Application”. *Journal of Modern Applied Statistical Methods*. 3: 85-103.

18. Jones, M.C. 2004. “Families of Distribution arising from Distribution of Order. *Statistic Test*. 13:1-43.

19. Kareema, A.K. and A.B. Mohammed. 2013. “Exponential Pareto Distribution”. *Journal of Mathematical Theory and Modeling*. 3(5):135-146.

20. Laroche, M., J. Bergeon, and G. Barbaro-Forleo. 2001. “Targeting Consumers who are willing to pay for Environmentally Friendly Products”. *Journal of Consumer Marketing*. 18(6): 503-520.

21. Mahenc, P. 2008. “Signaling the Environment Performance of Polluting Products to Green Consumers”. *International Journal of Industrial Organization*. 26:59-68.

22. Mahmoud, M.R. and R.M. Mandouh. 2013. “Transmuted Frechet”. *Journal of Applied Science Research*. 9(10):555-561.

23. Malhotra, N.K. and J.D. McCort. 2001. “A Cross-Cultural Comparison of Behavioral Intention Models: Theoretical Consideration and an Empirical Investigation”. *International Marketing Review*. 18(3): 235-269.

24. Marcelo, B., G. Indranil, and G.M. Cordeiro. 2016. “General Results for the Transmuted Family of Distributions and New Models”. *Journal of Probability and Statistics*.

25. Merovci, F. and L. Puka. 2014. “Transmuted Pareto Distribution”. *Probstat Forum*. 7: 1-11.

26. Mohammad, D. and A. Muhammad. 2014. “On the Mixture of BurrXII and Weibull Distribution”. *Journal of Statistics Application and Probability*. 3(2): 251-267.

27. Peattie, K. 2001. “Golden Goose or Wild Goose? The Hunt for Green Consumer”. *Business Strategy and Environment*. 10:187-199.

28. Shittu, O.I. and A.K. Adepoju. 2013. “On the Beta-Nakagami Distribution”. *Progress in Applied Mathematics*. 5:49-55.

29. Shaw, W. and I. Buckley. 2007. “The Alchemy of Probability Distributions: Beyond Gram-Charlier Expansions, and a Skew-Kurtotic-Normal Distribution from a Rank Transmutation Map”. Research Report. 2007.

30. Rowan, C. 2000. “Packaging by Design”. *Food Engineering International*. 19-25.

SUGGESTED CITATION

Akomolafe, A.A. and A. Maradesa. 2019. “On the Doubly Truncated Exponential Pareto Distribution”. *Pacific Journal of Science and Technology*. 20(1):103-111.

Pacific Journal of Science and Technology